JOURNAL OF ALGEBRA 129, 320–366 (1990)

The Character Table of the Hecke Algebra $\mathscr{H}(\mathrm{GL}_{2n}(\mathbf{F}_q), \mathrm{Sp}_{2n}(\mathbf{F}_q))$

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Communicated by Nagayoshi Iwahori

Received September 30, 1988

DEDICATED TO PROFESSOR SHINGO MURAKAMI ON HIS SIXTIETH BIRTHDAY

INTRODUCTION

In this paper we show how to calculate the character table of the Hecke algebra in the title. In particular, we shall see that it is "almost" identical with the table obtained by just replacing q by q^2 in the (suitably normalized) character table of the finite general linear group $\mathrm{GL}_n(\mathbf{F}_q)$.

Let $\mathbf{F}_q$ be the finite field of q elements, G the general linear group $\mathrm{GL}_{2n}(\mathbf{F}_q)$, and K the symplectic group $\mathrm{Sp}_{2n}(\mathbf{F}_q)$. We denote by V the complex G-module induced from the trivial K-module. Then it is known (Klyachko [9]) that the corresponding Hecke algebra $\mathscr{H}(G, K)$, which is

* Supported in part by NSF Grant DMS-8703075.

† Supported in part by the Grant-in-Aid for Scientific Research, the Ministry of Education, Science and Culture.

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0021-8693/90 $3.00

isomorphic to the endomorphism algebra $\mathrm{End}_G(V)$, is commutative, and that its natural basis (indexed by $K\backslash G/K$) is in natural one-to-one correspondence with the set of conjugate classes of the finite general linear group $\mathrm{GL}_n(\mathbf{F}_q)$. A conjecture [1] due to two of the authors (Bannai and Song) says that if one replaces q by q^2 in the values of an irreducible character of $\mathrm{GL}_n(\mathbf{F}_q)$ whose degree is prime to q, then one gets, up to a normalization, the values of an irreducible character of $\mathscr{H}(G, K)$. (See also [2], where, among other things, a similar relation between the irreducible characters of Paige's simple Moufang loop $\mathrm{M}(\mathbf{F}_q)$ and those of the special linear group $\mathrm{SL}_2(\mathbf{F}_q)$ was established.) Concerning the irreducible characters of $\mathrm{GL}_n(\mathbf{F}_q)$, we have a beautiful theory due to J. A. Green [7] (see also [11]). The purpose of the present paper is to develop a character theory of $\mathscr{H}(G, K)$ which is in some sense similar to that of Green. In particular, we shall see that the above-mentioned conjecture is true.

The paper is organized as follows. In Section 1, we recall some known results on conjugate classes and irreducible characters of finite general linear groups. In Section 2, we review general results on commutative Hecke algebras and Klyachko's results on the Hecke algebra $\mathscr{H}(G, K)$. We also prove an important result (Proposition 2.3.6) on the sizes of K-double cosets in G. In Section 3, we define the *basic function* $\chi_T^{\mathscr{H}}[\theta]$ on $\mathscr{H}(G, K)$ for an arbitrary pair (T, θ) of a maximal torus T of $\mathrm{GL}_n(\mathbf{F}_q)$ and a character θ of T, and prove (Lemma 3.3.3) that $\chi_T^{\mathscr{H}}[\theta]$ is a character of $\mathscr{H}(G, K)$ if θ is "in general position." In Section 4, we give the irreducible decomposition (Theorem 4.1.1) of the induced character $(1_K)^G$. The result shows that there exists a nice one-to-one correspondence between the irreducible characters of $\mathrm{GL}_n(\mathbf{F}_q)$ and those of $\mathscr{H}(G, K)$. In Section 5, we prove a formula (Theorem 5.3.2) expressing the values of basic functions in terms of Green polynomials of $\mathrm{GL}_n(\mathbf{F}_q)$. In Section 6, we give a simple algorithm by which one can write down the characters of $\mathscr{H}(G, K)$ as explicit linear combinations of basic functions. This, combined with the above-mentioned result in Section 5, enables us to calculate the character table of $\mathscr{H}(G, K)$ explicitly. See Theorem 6.6.1.

Notation. Let n be a natural number. For a power q of a prime p, we denote by n_q and $n_{q'}$ the p-part and p'-part of n, respectively, i.e., n_q is the highest power of p dividing n and $n_{q'} = n \cdot n_q^{-1}$. For a set S, $|S|$ denotes the cardinality of S. If τ is a transformation of S, S_τ denotes the set of τ-fixed points of S. If f is a function on S, and T is a subset of S, $f|T$ is the restriction of f to T. If G is a group, H is a subgroup of G, and x is an element of G, then $N_G(H)$, $Z_H(x)$, and $\mathrm{Cl}_H(x)$ denote the normalizer of H in G, the centralizer of x in H, and the H-conjugate class of x, respectively. As usual, $\mathbf{C}$, $\mathbf{R}$, $\mathbf{Q}$, and $\mathbf{Z}$ denote the sets of complex numbers, real numbers, rational numbers, and rational integers, respectively.

1. Preliminaries on General Linear Groups

Let k be the algebraic closure of a finite field $\mathbf{F}_q$ of q elements. We denote by $\mathbf{A}$ the general linear group $GL_n(k)$ over k, and by $A = \mathbf{A}(\mathbf{F}_q)$ the one over $\mathbf{F}_q$.

1.1. Here we collect some well-known results on the conjugate classes of the finite general linear group A.

A partition is a finite non-increasing sequence $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ of non-negative integers λ_i, called parts of λ. We consider $(\lambda_1, \lambda_2, ..., \lambda_r)$ and $(\lambda_1, \lambda_2, ..., \lambda_r, 0)$ to be the same partition. The length $|\lambda|$ of λ is defined by $|\lambda| = \sum_i \lambda_i$. Let $\mathscr{P}$ be the set of partitions. For $n = 0, 1, 2, ...,$ we put

$$\mathscr{P}_n = \{\lambda \in \mathscr{P};\ |\lambda| = n\}.$$

Let σ be the qth power map $t \to t^q$ of $k = \bar{\mathbf{F}}_q$ into itself. We denote by $O(M)$ the set of σ-orbits in the multiplicative group M of k. For $\alpha \in O(M)$, the length of the σ-orbit α is denoted by $|\alpha|$. We shall denote by $\mathscr{P}_{n,q}$ the set of $\mathscr{P}$-valued functions Λ on $O(M)$ such that

$$\sum_{\alpha \in O(M)} |\alpha| \cdot |\Lambda(\alpha)| = n.$$

We imbed $\mathscr{P}_n$ into $\mathscr{P}_{n,q}$ by putting

$$\lambda(\alpha) = \begin{cases} \lambda & \text{if} \quad \alpha = \{1\}, \\ (0) & \text{otherwise} \end{cases}$$

for $\lambda \in \mathscr{P}_n$ and $\alpha \in O(M)$. The conjugate (resp. unipotent conjugate) classes of A are in natural 1–1 correspondence with $\mathscr{P}_{n,q}$ (resp. $\mathscr{P}_n$). See [7, 11]. (For $n = 0$, we understand that $GL_n(\mathbf{F}_q) = \{1\}$.) For an element Λ of $\mathscr{P}_{n,q}$, let a_Λ be a representative of the corresponding conjugate class of A. If $(a_\Lambda)_s$ is the semisimple part of a_Λ, its conjugate class corresponds to $\Lambda_s \in \mathscr{P}_{n,q}$ defined by

$$\Lambda_s(\alpha) = (1^{|\Lambda(\alpha)|}), \qquad \alpha \in O(M).$$

The centralizer $Z_A((a_\Lambda)_s)$ of $(a_\Lambda)_s$ has the following form:

$$Z_A((a_\Lambda)_s) \simeq \prod_{\alpha \in O(M)} GL_{|\Lambda(\alpha)|}(\mathbf{F}_{q^{|\alpha|}}). \tag{1.1.1}$$

Under an isomorphism (1.1.1) the unipotent part $(a_\Lambda)_u$ of a_Λ goes to a unipotent element of $\prod_{\alpha \in O(M)} GL_{|\Lambda(\alpha)|}(\mathbf{F}_{q^{|\alpha|}})$ whose class corresponds to

$(\Lambda(\alpha))_{\alpha\in O(M)}\in\prod_{\alpha\in O(M)}\mathscr{P}_{|\Lambda(\alpha)|}$. For an element $\lambda=(1^{r_1(\lambda)}2^{r_2(\lambda)}\cdots)$ of $\mathscr{P}_n$, we have (see [17, IV])

$$Z_A(a_\lambda)\simeq\prod_{i\geqslant 1}\mathrm{GL}_{r_i(\lambda)}(\mathbf{F}_q)\ltimes\mathbf{U}(\mathbf{F}_q), \tag{1.1.2}$$

where $\mathbf{U}(\mathbf{F}_q)$ is a unipotent subgroup whose order is equal to $q^{d(\lambda)}$, $d(\lambda)=\sum_i(r_i(\lambda)+r_{i+1}(\lambda)+\cdots)^2-\sum_i r_i(\lambda)^2$. By (1.1.1) and (1.1.2), we have

$$|\mathrm{Cl}_A(a_\Lambda)|=|\mathrm{GL}_n(\mathbf{F}_q)|\prod_{\alpha\in O(M)}q^{-|\alpha|\,d(\Lambda(\alpha))}\cdot\left|\prod_i\mathrm{GL}_{r_i(\Lambda(\alpha))}(\mathbf{F}_{q^{|\alpha|}})\right|^{-1} \tag{1.1.3}$$

for $\Lambda\in\mathscr{P}_{n,q}$. For a fixed Λ, let $\{\alpha_1,\ldots,\alpha_m\}$ be the elements of $O(M)$ such that $\Lambda(\alpha_i)\neq(0)$. Then the number (1.1.3) can be considered as a polynomial in q whose coefficients depend only on $|\alpha_i|$ and $\Lambda(\alpha_i)$, $1\leqslant i\leqslant m$.

1.2. Here we review Green's theory [7] on the irreducible characters of $A=\mathrm{GL}_n(\mathbf{F}_q)$ in a form convenient for our purpose.

Let $k=\bar{\mathbf{F}}_q$ and $\sigma\colon k\to k$ the qth power map as in 1.1. Then σ acts on $\mathbf{A}$ so that $A=\mathbf{A}_\sigma$. Let $a\in\mathrm{Cl}_A(a_\Lambda)$ $(\Lambda\in\mathscr{P}_{n,q})$ be a fixed semisimple element of A. We put

$$\mathbf{C}=Z_{\mathbf{A}}(a)$$

and

$$C=\mathbf{C}_\sigma\simeq\prod_{\alpha\in O(M)}\mathrm{GL}_{|\Lambda(\alpha)|}(\mathbf{F}_{q^{|\alpha|}})$$

(see (1.1.1)). Let $\mathbf{T}_1$ be a maximally split σ-stable maximal torus of the algebraic group $\mathbf{C}$. Then

$$T_1=(\mathbf{T}_1)_\sigma\simeq\prod_{\alpha\in O(M)}(\mathbf{F}^{\times}_{q^{|\alpha|}})^{|\Lambda(\alpha)|}.$$

We define the Weyl group $W(C)$ of C by

$$W(C)=N_C(\mathbf{T}_1)/T_1\simeq\prod_{\alpha\in O(M)}S_{|\Lambda(\alpha)|}.$$

where S_m denotes the symmetric group of degree m. Then [17, Chap. II] the C-conjugate classes of σ-stable maximal tori of $\mathbf{C}$ are in natural 1–1 correspondence with the conjugate classes of $W(C)$. If $\mathbf{T}$ is a σ-stable maximal torus whose class corresponds to that of $w\in W(C)$, we write $\mathbf{T}$ as $\mathbf{T}_w$. We also write

$$T_w=(\mathbf{T}_w)_\sigma.$$

Such T_w's are called maximal tori of C. For example, if $\mathbf{C} = \mathbf{A}$ (hence $W(C) \simeq S_n$), and $w \in S_n$ is of cycle type $(e_1, e_2, ...)$, then

$$T_w \simeq \prod_i \mathbf{F}_{q^{e_i}}^{\times}.$$

Let $V(C)$ be the set of unipotent elements of C. For $u, v \in V(C)$, we write

$$\mathrm{Cl}_C(u) \leqslant \mathrm{Cl}_C(v) \tag{1.2.1}$$

if the Zariski-closure of $\mathrm{Cl}_{\mathbf{C}}(v)$ contains u. The unipotent conjugate classes of C are parametrized by

$$\mathscr{P}_C = \prod_{\alpha \in O(M)} \mathscr{P}_{|A(\alpha)|}.$$

If the class of $u \in V(C)$ corresponds to $((|A(\alpha)|))_\alpha \in \mathscr{P}_C$, then $\mathrm{Cl}_C(u)$ is the unique maximal class with respect to the ordering (1.2.1). Such a u is called a regular unipotent element of C. (See, e.g., [4, 13.4] for a well-known interpretation of the ordering (1.2.1) in terms of partitions.) If $u \in V(C)$ is in the class corresponding to $(\lambda_\alpha)_\alpha \in \mathscr{P}_C$, we define φ_u to be the irreducible character $\bigotimes_\alpha \varphi_{\lambda_\alpha}$ of $W(C) \simeq \prod_\alpha S_{|A(\alpha)|}$. For example, if u is regular unipotent, φ_u is the identity character. For a maximal torus T of C, let $Q_T^C(\cdot)$ be the *Green function* [11, 6, 16] of C. The Green functions $\{Q_T^C\}$ are real valued functions on $V(C)$ satisfying conditions 1.2.2–1.2.5 below. Conversely, these four conditions determine $\{Q_T^C\}$ uniquely.

1.2.2. $Q_T^C(u) = Q_T^C(u')$ if $\mathrm{Cl}_C(u) = \mathrm{Cl}_C(u')$.

1.2.3. $Q_T^C(\cdot) = Q_{T'}^C(\cdot)$ if T and T' are C-conjugate.

1.2.4. For a fixed $u \in V(C)$, let $Q_\bullet^C(u)$ be the class function $w \to Q_{T_w}^C(u)$ on $W(C)$. Then $Q_\bullet^C(u) - q^{d(u)}\varphi_u$ is a real linear combination of $Q_\bullet^C(v)$'s such that

$$\mathrm{Cl}_C(u) \leqslant \mathrm{Cl}_C(v).$$

Here $d(u)$ is the non-negative integer defined by

$$d(u) = (\dim Z_{\mathbf{C}}(u) - n)/2.$$

In particular, if u_{reg} is a regular unipotent element of C, then

$$Q_\bullet^C(u_{\mathrm{reg}}) = \varphi_{u_{\mathrm{reg}}} = \mathrm{id}_{W(C)}.$$

1.2.5. For $u, v \in V(C)$, we have

$$|W(C)|^{-1} \sum_{w \in W(C)} |T_w|\, Q_{T_w}^C(u)\, Q_{T_w}^C(v) = 0$$

if $\mathrm{Cl}_C(u) \neq \mathrm{Cl}_C(v)$.

In fact, that $Q_T^C(\cdot)$'s satisfy these conditions can be seen, e.g., in [11, III; 7] or in [16]. On the other hand, using 1.2.4 and 1.2.5, we can show the uniqueness of the set $\{Q_\bullet^C(u);\ u \in V(C) \text{ modulo conjugacy}\}$ of class functions on $W(C)$ by downward induction with respect to the ordering (1.2.1). It is also known that there exists a set of polynomials ("*Green polynomials*") $\{Q_w^C((\lambda_\alpha)_\alpha);\ w \in W(C),\ (\lambda_\alpha)_\alpha \in \mathscr{P}_C\}$ in an indeterminate t with integer coefficients independent of q such that

$$Q_{T_w}^C(u) = Q_w^C((\lambda_\alpha)_\alpha)(q)$$

if the class of u corresponds to $(\lambda_\alpha)_\alpha \in \mathscr{P}_C$.

Let T be a maximal torus of C, and θ a character of T. We define a class function $\zeta_T^C[\theta]$ on C by

$$\zeta_T^C[\theta](su) = \sum_{\substack{x \in C/Z(s) \\ xsx^{-1} \in T}} Q_{x^{-1}Tx}^{Z(s)}(u)\, \theta(xsx^{-1}) \tag{1.2.6}$$

(see [7, Def. 4.11; 6, Th. 4.2]), where s is a semisimple element of C, u a unipotent element of $Z(s) = Z_C(s)$. (Here we identified $C/Z(s)$ with a set of representatives of it.) Then we know [7, 6] that $\zeta_T^C[\theta]$ is a virtual character of C, and that

$$\begin{aligned} \zeta_T^C[\theta](1) = Q_T^C(1) &= \varepsilon(C)\, \varepsilon(T)\, |C/T|_{q'} \\ &= \varepsilon(C)\, \varepsilon(T)\, |C/T|\, q^{-N(C)}, \end{aligned} \tag{1.2.7}$$

where $\varepsilon(\mathbf{D}_\sigma) = (-1)^{s(\mathbf{D})}$ ($s(\mathbf{D})$ = the split rank of D) for any algebraic $\mathbf{F}_q$-group $\mathbf{D}$, and

$$N(C) = (\dim \mathbf{C} - n)/2.$$

We also know that any irreducible character of C can be written as a rational linear combination of $\zeta_T^C[\theta]$'s. To state this result more explicitly in the case $C = A$, we need some notations. For each positive integer m, let M_m be the multiplicative group of $\mathbf{F}_{q^m}$. If l divides m, the norm mapping $M_m \to M_l$ gives a surjective homomorphism, hence defines an injective homomorphism $\operatorname{Hom}(M_l, \mathbf{C}^\times) \to \operatorname{Hom}(M_m, \mathbf{C}^\times)$. We define an infinite discrete group L by

$$L = \varinjlim \operatorname{Hom}(M_m, \mathbf{C}^\times).$$

The qth power map σ acts on L. We have

$$(L)_{\sigma^m} = \operatorname{Hom}(M_m, \mathbf{C}^\times)$$

(see [11, IV, 1]). Let $O(L)$ be the set of σ-orbits in L. For $\alpha \in O(L)$, the length of α is denoted by $|\alpha|$. We shall denote by $\hat{\mathscr{P}}_{n,q}$ the set of partition-valued function Ψ on $O(L)$ such that

$$\sum_{\alpha \in O(L)} |\alpha| \, |\Psi(\alpha)| = n.$$

For $\alpha \in O(L)$, let θ_α be an element of $\mathrm{Hom}(M_{|\alpha|}, \mathbf{C}^\times)$ such that

$$\alpha = \{\theta_\alpha, \theta_\alpha \circ \sigma, ...\}. \tag{1.2.8}$$

Then, for $\Psi \in \hat{\mathscr{P}}_{n,q}$, $\theta_\alpha \circ \det$ is a (1-dimensional) character of $\mathrm{GL}_{|\Psi(\alpha)|}(\mathbf{F}_{q^{|\alpha|}})$, and

$$\theta_\Psi = \prod_\alpha \theta_\alpha \circ \det$$

is a character of

$$C_\Psi = \prod_\alpha \mathrm{GL}_{|\Psi(\alpha)|}(\mathbf{F}_{q^{|\alpha|}}).$$

We can identify C_Ψ with the centralizer of a semisimple element of A (see (1.1.1)). Let φ_Ψ be the irreducible character

$$\bigotimes_\alpha \varphi_{\Psi(\alpha)}$$

of the Weyl group

$$W(\Psi) = W(C_\Psi) \simeq \prod_\alpha S_{|\Psi(\alpha)|}$$

of C_Ψ. We put

$$\zeta_\Psi = |W(\Psi)|^{-1} \sum_{w \in W(\Psi)} \varphi_\Psi(w) \, \zeta^A_{T_w}[\theta_\Psi \mid T_w], \tag{1.2.9}$$

where each T_w ($w \in W(\Psi)$) is chosen so that it lies in C_Ψ. This function does not depend on the choice of θ_α in (1.2.8).

1.2.10. THEOREM [7]. *If $\Psi \in \hat{\mathscr{P}}_{n,q}$, ζ_Ψ is an irreducible character of $A = \mathrm{GL}_n(\mathbf{F}_q)$ up to sign. The correspondence $\Psi \to \pm\zeta_\Psi$ is a bijection between $\hat{\mathscr{P}}_{n,q}$ and the set of irreducible characters of A.*

1.2.11. For $\Psi \in \hat{\mathscr{P}}_{n,q}$, we define $\Psi_s \in \hat{\mathscr{P}}_{n,q}$ by

$$\Psi_s(\alpha) = (|\Psi(\alpha)|), \qquad \alpha \in O(L).$$

Let $\zeta = \pm\zeta_{\Psi}$ ($\Psi \in \hat{\mathscr{P}}_{n,q}$) be an irreducible character of A. The irreducible character ζ_s of A defined by

$$\zeta_s = \pm\zeta_{\Psi_s}$$

is called the semisimple part of ζ. Note that $(\zeta_s)_s = \zeta_s$. An irreducible character of A is said to be semisimple, if it is equal to its semisimple part. This definition agrees with the one given in [4, 8.4]. In particular, an irreducible character $\pm\zeta_{\Psi}$ of A is semisimple if and only if its degree is prime to q. In that case, the degree of $\pm\zeta_{\Psi}$ is equal to $|A/C_{\Psi}|_{q'}$. In general, the degree $\zeta(1)$ of an irreducible character $\zeta = \pm\zeta_{\Psi}$ is equal to $\zeta_s(1)$ times the degree of a unipotent character of C_{Ψ} (see [7, Th. 14; 4, 12.9]).

1.2.12. Let G be the finite general linear group $\mathrm{GL}_{2n}(\mathbf{F}_q)$, and Q the parabolic subgroup of G given by

$$Q = \left\{ \begin{pmatrix} b & c \\ 0 & a \end{pmatrix} \in G;\, a, b \in A \right\}.$$

For an irreducible character ζ of A, let $\zeta \otimes \zeta$ denote the one of Q defined by

$$(\zeta \otimes \zeta) \begin{pmatrix} b & c \\ 0 & a \end{pmatrix} = \zeta(a)\,\zeta(b), \qquad \begin{pmatrix} b & c \\ 0 & a \end{pmatrix} \in Q.$$

1.2.13. Lemma. *Let $\zeta = \pm\zeta_{\Psi}$ ($\Psi \in \hat{\mathscr{P}}_{n,q}$) be an irreducible character of A, and $\pm\zeta_{\Omega}$ ($\Omega \in \hat{\mathscr{P}}_{2n,q}$) that of G contained in the induced character $(\zeta \otimes \zeta)^G$.*

(i) *The semisimple part $\pm\zeta_{\Omega_s}$ of $\pm\zeta_{\Omega}$ is uniquely determined from that of ζ. More explicitly, we have*

$$\Omega_s(\alpha) = (2\,|\Psi(\alpha)|), \qquad \alpha \in O(L).$$

In particular, $\Omega(\alpha) \in \mathscr{P}_{2|\Psi(\alpha)|}$ for any $\alpha \in O(L)$.

(ii) *For any $\alpha \in O(L)$, we have*

$$\Omega(\alpha) \leqslant (\Psi(\alpha)', \Psi(\alpha)')',$$

where, for $\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathscr{P}_m$, $\lambda' = (\lambda'_1, \lambda'_2, \ldots) \in \mathscr{P}_m$ denotes the dual partition of λ,

$$\lambda'_i = |\{j;\, \lambda_j \geqslant i\}|,$$

and (λ, λ) is an element of $\mathscr{P}_{2m}$ given by

$$(\lambda, \lambda) = (\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots),$$

and the ordering $\leqslant$ is the natural ordering [11] on $\mathscr{P}_{|\Omega(\alpha)|}$.

Proof. Let T be a maximal torus of A, and θ a character of T. Then

$$T\times T=\left\{\begin{pmatrix} t_1 & 0\\ 0 & t_2\end{pmatrix}\in G;\, t_i\in T\right\}$$

is a maximal torus of G, and

$$\theta\otimes\theta\colon \begin{pmatrix} t_1 & 0\\ 0 & t_2\end{pmatrix}\to\theta(t_1)\,\theta(t_2),\qquad t_i\in T$$

is a character of $T\times T$. By [7, Lem. 5.1], we have

$$(\zeta_T^A[\theta]\otimes\zeta_T^A[\theta])^G=\zeta_{T\times T}^G[\theta\otimes\theta].$$

Hence, from (1.2.9), we see that $(\zeta_\Psi\otimes\zeta_\Psi)^G$ is equal to

$$|W(\Psi,\Psi)|^{-1}\sum_{w\in W(\Psi,\Psi)}(\varphi_\Psi\otimes\varphi_\Psi)^{W(\Psi,\Psi)}(w)\,\zeta_{T_w}^G[\theta_{(\Psi,\Psi)}\,|\,T_w],$$

where (Ψ,Ψ) is an element of $\hat{\mathscr{P}}_{2n,q}$ defined by

$$(\Psi,\Psi)(\alpha)=(\Psi(\alpha),\Psi(\alpha)),\qquad \alpha\in O(L).$$

Part (i) of the lemma follows from this and (1.2.9). To prove (ii), we have only to know which irreducible characters of $W(\Psi,\Psi)$ are contained in $(\varphi_\Psi\otimes\varphi_\Psi)^{W(\Psi,\Psi)}$. But, if we define $\Psi'\in\hat{\mathscr{P}}_{n,q}$ by

$$\Psi'(\alpha)=\Psi(\alpha)',\qquad \alpha\in O(L),$$

we have

$$(\varphi_\Psi\otimes\varphi_\Psi)^{W(\Psi,\Psi)}=((\varphi_{\Psi'}\otimes\varphi_{\Psi'})^{W(\Psi,\Psi)})\otimes\mathrm{sgn}_{W(\Psi,\Psi)},$$

where $\mathrm{sgn}_{W(\Psi,\Psi)}$ is the sign character of $W(\Psi,\Psi)$. Hence an irreducible character φ_Ω ($\Omega\in\hat{\mathscr{P}}_{2n,q}$, $\Omega_s=(\Psi,\Psi)_s$) of $W(\Psi,\Psi)$ is contained in $(\varphi_\Psi\otimes\varphi_\Psi)^{W(\Psi,\Psi)}$ if and only if $\varphi_{\Omega'}$ is contained in $(\varphi_{\Psi'}\otimes\varphi_{\Psi'})^{W(\Psi,\Psi)}$. By a well-known criterion, this happens only if

$$\Omega'(\alpha)\geqslant(\Psi'(\alpha),\Psi'(\alpha)),\qquad \alpha\in O(L),$$

which is equivalent to

$$\Omega(\alpha)\leqslant(\Psi'(\alpha),\Psi'(\alpha))',\qquad \alpha\in O(L),$$

This proves the lemma.

1.3. Let C be as in 1.2. For a maximal torus T of C, we define a function $\tilde{Q}_T^C$ on $V(C)$ by

$$\tilde{Q}_T^C(u) = |\mathrm{Cl}_C(u)|\, Q_T^C(u)\, Q_T^C(1)^{-1}, \qquad u \in V(C). \tag{1.3.1}$$

1.3.2. LEMMA. (i) *$\tilde{Q}_T^C(u) = \tilde{Q}_T^C(u')$ if $\mathrm{Cl}_C(u) = \mathrm{Cl}_C(u')$.*

(ii) *$\tilde{Q}_T^C(\cdot) = \tilde{Q}_{T'}^C(\cdot)$ if T and T' are C-conjugate.*

(iii) *For a fixed $u \in V(C)$, let $\tilde{Q}_{\cdot}^C(u)$ be the class function $w \to \tilde{Q}_{T_w}^C(u)$ on $W(C)$. Then $\tilde{Q}_{\cdot}^C(u) - q^{e(u)}\,\mathrm{sgn}_{W(C)} \otimes \varphi_u$ is a real linear combination of $\tilde{Q}_{\cdot}^C(v)$'s such that*

$$\mathrm{Cl}_C(v) \leqslant \mathrm{Cl}_C(u).$$

Here, φ_u is the character of $W(C)$ defined in 1.2, and

$$e(u) = \dim \mathrm{Cl}_C(u)/2.$$

In particular,

$$\tilde{Q}_{\cdot}^C(1) = \mathrm{id}_{W(C)}.$$

(iv) *For $u, v \in V(C)$, we have*

$$|W(C)|^{-1} \sum_{w \in W(C)} |T_w|^{-1}\, \tilde{Q}_{T_w}^C(u)\, \tilde{Q}_{T_w}^C(v) = 0$$

if $\mathrm{Cl}_C(u) \neq \mathrm{Cl}_{\mathscr{V}}(v)$.

(v) *Conditions* (i)–(iv) *determine the set $\{\tilde{Q}_T^C\}$ of functions on $V(C)$ uniquely.*

(vi) *For $w \in W(C)$ and $(\lambda_\alpha)_\alpha \in \mathscr{P}_C$, there exists a polynomial $\tilde{Q}_w^C((\lambda_\alpha)_\alpha)(t)$ with integer coefficients independent of q such that*

$$\tilde{Q}_{T_w}^C(u) = \tilde{Q}_w^C((\lambda_\alpha)_\alpha)(q) \tag{1.3.3}$$

if the class of $u \in V(C)$ corresponds to $(\lambda_\alpha)_\alpha$.

Proof. (i), (ii), (iv) By the definition (1.3.1) of $\tilde{Q}_T^C$, we can prove these easily using 1.2.2, 1.2.3, and 1.2.5, respectively.

(iii) In addition to 1.2.5, we know (see [11, III, 7])

$$|W(C)|^{-1} \sum_{w \in W(C)} |T_w|\, Q_{T_w}^C(u)^2 = |Z_C(u)| \tag{1.3.4}$$

for $u \in V(C)$. By 1.2.4, 1.2.5, and (1.3.4), we see that the class function

$$f_u : w \to |T_w|\, Q_{T_w}^C(u) - q^{-d(u)}\, |Z_C(u)|\, \varphi_u(w)$$

on $W(C)$ can be written as a linear combination of $\{\varphi_v;\ \mathrm{Cl}_C(v) \geqslant \mathrm{Cl}_C(u)\}$. Hence the function $\mathrm{sgn}_{W(C)} \otimes f_u$ can be written as a linear combination of $\{\alpha_{v'};\ \mathrm{Cl}_C(v') \leqslant \mathrm{Cl}_C(u')\}$. Here u' is an element of $V(C)$ such that $\varphi_{u'} = \mathrm{sgn}_{W(C)} \otimes \varphi_u$. On the other hand, by (1.2.7) and (1.3.1),

$$\tilde{Q}^C_{T_w}(u) = \mathrm{sgn}_{W(C)}(w)\ |T_w|\ Q^C_{T_w}(u)\ |Z_C(u)|^{-1}\ q^{N(C)}$$

for $u \in V(C)$. Hence the function $\tilde{Q}^C_{\cdot}(u)$ on $W(C)$ can be written as a linear combination of $\{\varphi_{v'};\ \mathrm{Cl}_C(v') \leqslant \mathrm{Cl}_C(u')\}$, and the coefficient of $\varphi_{u'}$ is

$$q^{N(C)-d(u)} = q^{(\dim C - \dim Z_C(u))/2} = q^{e(u)}.$$

Hence (iii) follows.

(v) This can be shown by upward induction with respect to the ordering (1.2.1).

(vi) It is easy to see the (unique) existence of rational functions $\tilde{Q}^C_w((\lambda_\alpha)_\alpha)(t)$ satisfying (1.3.3). Assume that q is large so that there exists a θ such that $\zeta^C_T[\theta]$ is irreducible up to sign. Then, for any $x \in C$, $\tilde{\zeta}^C_T(x) = |\mathrm{Cl}_C(x)|\ \zeta^C_T[\theta](x) \cdot \zeta^C_T[\theta](1)^{-1}$ is an algebraic integer. In particular for any $u \in V(C)$, $\tilde{Q}^C_T(u) = \tilde{\zeta}^C_T(u)$ is a rational integer. Hence, by (1.3.3), $\tilde{Q}^C_w((\lambda_\alpha)_\alpha)(t)$ must be a polynomial. Since $|\mathrm{Cl}_C(u)|$ and $Q^C_T(u)$ can be considered as polynomials in q with integer coefficients, and $Q^C_T(1)$ is a monic polynomial in q, the coefficients of $\tilde{Q}^C_w((\lambda_\alpha)_\alpha)$ must be integers. This proves (vi).

1.3.5. *Remark.* The above lemma and its counterpart for Q^C_T mentioned in 1.2 can be formulated so that they are true for any finite reductive group C.

2. Preliminaries on Hecke Algebras

2.1. Let G be a finite group, and $\mathbf{C}G$ the complex group algebra of G. The character $(1_K)^G$ of G induced from the trivial character of a subgroup K is afforded by the G-module $\mathbf{C}Ge$ with $e = e(K) = |K|^{-1} \sum_{k \in K} k$. The *Hecke algebra* $\mathscr{H} = \mathscr{H}(G, K)$ is, by definition, the subalgebra $e \cdot \mathbf{C}G \cdot e$ of $\mathbf{C}G$. For general facts concerning Hecke algebras, we refer the reader to [5, Sect. 11D]. From now on, we assume that $\mathscr{H}$ is commutative, or equivalently, that $(1_K)^G$ is multiplicity-free. Let $\eta_1, \eta_2, ..., \eta_d$ ($d = \dim_{\mathbf{C}} \mathscr{H}$) be the irreducible characters of G contained in $(1_K)^G$. We extend each η_i to a $\mathbf{C}$-linear funtion on $\mathbf{C}G$. Then the restrictions $\eta_i | \mathscr{H}$, $1 \leqslant i \leqslant d$, are the 1-dimensional representations, or the irreducible characters, of $\mathscr{H}$. In particular,

$$\eta_i(e) = 1, \qquad 1 \leqslant i \leqslant d.$$

Let

$$G = \bigcup_{j=1}^{d} Ka_jK \qquad \text{(disjoint)} \tag{2.1.1}$$

be the K-double cosets decomposition of G. For $x \in G$, we put

$$\operatorname{ind} x = H - \operatorname{ind} x = |K \times K/K| = |K/(K \cap x^{-1}Kx)|$$

and

$$[x] = \operatorname{ind} x \cdot exe \qquad (\in \mathscr{H}).$$

The d-by-d matrix $(\eta_i([a_j]))$, which is independent of the choice of $\{a_j\}$ in (2.1.1), is called the *character table* of the Hecke algebra $\mathscr{H}$. The primitive idempotent ε_{η_i} of $\mathscr{H}$ affording η_i is given by

$$\varepsilon_{\eta_i} = \eta_i(1)\,|G/K|^{-1} \sum_{j=1}^{d} \eta_i([a_j^{-1}])\, ea_je. \tag{2.1.2}$$

This implies that the $\eta_i([a_j])$'s satisfy the following orthogonality relations:

$$\begin{aligned}&\eta_i(1)\,|G/K|^{-1} \sum_{j=1}^{d} (\operatorname{ind} a_j)^{-1}\, \eta_i([a_j])\, \eta_{i'}([a_j^{-1}])\\ &= \begin{cases} 1 & \text{if } i = i'; \\ 0 & \text{if } i \neq i'. \end{cases}\end{aligned} \tag{2.1.3}$$

2.2. Let τ be an automorphism of a finite group G. We put

$$K = G_\tau = \{x \in G;\, x^\tau = x\}.$$

The following lemma is easy.

2.2.1. LEMMA. (i) *For x, $y \in G$, we have $KxK = KyK$ if and only if $xx^{-\tau}$ (resp. $x^{-\tau}x$) and $yy^{-\tau}$ (resp. $y^{-\tau}y$) are K-conjugate. Here $x^{-\tau}$ stands for $(x^{-1})^\tau$.*

(ii) *For $x \in G$, let* $\operatorname{ind} x = H - \operatorname{ind} x$ *be as in* 2.1. *Then we have*

$$\operatorname{ind} x = |K/Z_K(xx^{-\tau})| = |K/Z_K(x^{-\tau}x)|.$$

(iii) *Assume that τ is involutive. Then $(xx^{-\tau})^\tau = (xx^{-\tau})^{-1}$. In particular, $Z_G(xx^{-\tau})$ is τ-stable, and $Z_K(xx^{-\tau}) = (Z_G(xx^{-\tau}))_\tau$.*

2.3. Now we take G to be the general linear group $\mathrm{GL}_{2n}(\mathbf{F}_q)$ of degree $2n$ over a finite field $\mathbf{F}_q$. We define the involutive automorphism τ of G by

$$x^\tau = J^{-1}({}^t x)^{-1} J, \qquad x \in G, \tag{2.3.1}$$

where ${}^t x$ is the transposed matrix of x and

$$J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$$

with 1_n the identity matrix of degree n. We put

$$K = G_\tau \simeq \mathrm{Sp}_{2n}(\mathbf{F}_q), \tag{2.3.2}$$

and

$$A = \left\{ \begin{pmatrix} 1_n & 0 \\ 0 & x \end{pmatrix} \in G;\ x \in \mathrm{GL}_n(\mathbf{F}_q) \right\} \qquad (\simeq \mathrm{GL}_n(\mathbf{F}_q)) \tag{2.3.3}$$

If $n = 0$, we understand that $G = K = A = \{1\}$.

2.3.4. LEMMA (Klyachko [9]). *Let $\{a_\Lambda;\ \Lambda \in \mathscr{P}_{n,q}\}$ be a set of representatives of the conjugate classes of A (see* 1.1).

(i) *For $a \in A$, $KaK = Ka_\Lambda K$ if and only if a is conjugate to a_Λ.*

(ii) $G = \bigcup_{\Lambda \in P_{n,q}} Ka_\Lambda K.$

A standard argument (due to I. M. Gelfand) shows that the above lemma implies the following:

2.3.5. THEOREM (Klyachko [9]). *The Hecke algebra $\mathscr{H} = \mathscr{H}(G, K)$ is commutative. In other words, the induced character $(1_K)^G$ is multiplicity-free.*

In general, if S is an infinite set of powers of primes, and D is a quantity defined for $\mathbf{F}_q$, $q \in S$, and D can be expressed as a rational function in q with coefficients independent of q, then we shall write

$$D_{q \to q^2}$$

for the quantity obtained from D by the formal replacement $q \to q^2$. For example, we can consider $|\mathrm{Cl}_A(a_\Lambda)|_{q \to q^2}$, $\Lambda \in \mathscr{P}_{n,q}$ (see (1.1.3)). The main result of this subsection is:

2.3.6. PROPOSITION. *For $\Lambda \in \mathscr{P}_{n,q}$ we have*

$$\mathscr{H} - \mathrm{ind}\ a_\Lambda = |\mathrm{Cl}_A(a_\Lambda)|_{q \to q^2}.$$

Proof. By 2.2.1(ii) and the Jordan decomposition of a_Λ we get

$$\mathrm{ind}\ a_\Lambda = |K|\ |Z_{K(s)}(uu^{-\tau})|^{-1}, \tag{2.3.7}$$

where $s=(a_\Lambda)_s$, $u=(a_\Lambda)_u$, and $K(s)=Z_K(ss^{-\tau})$. In particular, when a_Λ is semisimple, i.e., when $\Lambda=\Lambda_s$ (see 1.1), we have

$$\operatorname{ind} a_\Lambda = |K|\, |Z_K(a_\Lambda(a_\Lambda)^{-\tau})|^{-1}. \tag{2.3.8}$$

Moreover, we can show

$$Z_K(a_\Lambda(a_\Lambda)^{-\tau}) \simeq \prod_{\alpha\in O(M)} \mathrm{Sp}_{2|\Lambda(\alpha)|}(\mathbf{F}_{q^{|\alpha|}}), \tag{2.3.9}$$

if $\Lambda=\Lambda_s$. To prove this, let $\mathbf{G}$ be the general linear group $\mathrm{GL}_{2n}(k)$ over $k=\bar{\mathbf{F}}_q$, and τ, $\mathbf{K}$, and $\mathbf{A}$ the objects defined as in (2.3.1), (2.3.2), and (2.3.3), respectively, by using k instead of $\mathbf{F}_q$. Then $a_\Lambda(a_\Lambda)^{-\tau}$ is $\mathbf{K}$-conjugate to a diagonal element. Hence, it is easy to see that

$$\begin{aligned} Z_{\mathbf{K}}(a_\Lambda(a_\Lambda)^{-\tau}) &= (Z_{\mathbf{G}}(a_\Lambda(a_\Lambda)^{-\tau}))_\tau \\ &\simeq \prod_{\alpha\in O(M)} (\mathrm{Sp}_{2|\Lambda(\alpha)|}(k))^{|\alpha|}. \end{aligned} \tag{2.3.10}$$

Since $Z_K(a_\Lambda(a_\Lambda)^{-\tau})=Z_{\mathbf{K}}(a_\Lambda(a_\Lambda)^{-\tau})_\sigma$, we get (2.3.9) by observing how the (qth power) action of σ on the left hand side of (2.3.10) is converted to that on the right hand side. We omit the details. By (2.3.8), (2.3.9), and (1.1.3), we see that the proposition holds when a_Λ is semisimple. Moreover, by (2.4.1) below, to prove the proposition for a general a_Λ, we can assume that a_Λ is unipotent, i.e, that $\Lambda=\lambda=(\lambda_1, \lambda_2, ..., \lambda_r)\in\mathscr{P}_n$. Then $X=a_\lambda(a_\lambda)^{-\tau}-1_{2n}$ is a nilpotent matrix, for which one can find a set $\{e_i, e_j';\ 1\leqslant i, j\leqslant s\}$ of vectors of $V=k^{2n}$ with the properties

$$X^{\lambda_i}e_i = X^{\lambda_i}e_i' = 0;$$

$\bigcup_{i,j}\{X^je_i,\ X^je_i';\ 0\leqslant j<\lambda_i-1\}$ is a basis of V;

$$\langle X^le_i, X^me_j'\rangle_J = \begin{cases} 1 & \text{if} \quad i=j \text{ and } l+m=\lambda_i-1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\langle X^le_i, X^me_j\rangle_J = \langle X^le_i', X^me_j'\rangle_J = 0,$$

where $\langle\cdot,\cdot\rangle_J$ is the skew-symmetric bilinear form on V associated with the matrix J. If Y is a linear transformation which commutes with X, then we can write

$$Ye_i = \sum_{j=1}^{s} \left\{ \sum_{\max(0,\lambda_j - \lambda_i) \leqslant h < \lambda_j} (c_{ijh} X^h e_j + c'_{ijh} X^h e'_j) \right\},$$

$$Ye'_i = \sum_{j=1}^{s} \left\{ \sum_{\max(0,\lambda_j - \lambda_i) \leqslant h < \lambda_j} (d_{ijh} X^h e_j + d'_{ijh} X^h e'_j) \right\},$$

with c_{ijh}, c'_{ijh}, d_{ijh}, $d'_{ijh} \in k$. Let $Z_{\mathbf{G}}(X)_r$ be the subgroup of $Z_{\mathbf{G}}(X)$ which consists of the elements $Y \in Z_{\mathbf{G}}(X)$ such that

$$Ye_i = \sum_{\{1 \leqslant j \geqslant s; \lambda_j = \lambda_i\}} (c_{ij0} e_j + c'_{ij0} e'_j)$$

and

$$Ye'_i = \sum_{\{1 \leqslant j \leqslant s; \lambda_j = \lambda_i\}} (d_{ij0} e_j + d'_{ij0} e'_j).$$

Then we have

$$Z_{\mathbf{G}}(X)_r \simeq \prod_{i \geqslant 1} \mathrm{GL}_{2r_i(\lambda)}(k)$$

if $\lambda = (1^{r_1(\lambda)}, 2^{r_2(\lambda)}, ...)$. We also have

$$Z_{\mathbf{G}}(a_\lambda (a_\lambda)^{-\tau}) = Z_{\mathbf{G}}(X) = Z_{\mathbf{G}}(X)_r \ltimes \mathbf{U},$$

where $\mathbf{U}$ is the unipotent radical of the algebraic group $Z_{\mathbf{G}}(X)$. See [17, IV, Sect. 1]. These results imply that

$$Z_{\mathbf{K}}(a_\lambda (a_\lambda)^{-\tau}) = Z_{\mathbf{K}}(X) = Z_{\mathbf{K}}(X)_r \ltimes (\mathbf{K} \cap \mathbf{U})$$
$$(Z_{\mathbf{K}}(X)_r = \mathbf{K} \cap Z_{\mathbf{G}}(X)_r),$$

and that

$$Z_{\mathbf{K}}(X)_r \simeq \prod_{i \geqslant 1} \mathrm{Sp}_{2r_i(\lambda)}(k).$$

Hence we get

$$|Z_K(a_\lambda (a_\lambda)^{-\tau})| = |Z_K(X)| = q^{c(\lambda)} \prod_{i \geqslant 1} |\mathrm{Sp}_{2r_i(\lambda)}(\mathbf{F}_q)|,$$

where $c(\lambda) = \dim \mathbf{K} \cap \mathbf{U}$. This formula, (2.3.7), and (1.1.2) imply that the equality in the proposition holds up to a power of q. Hence, to complete our proof it is enough to show that

$$\dim \mathbf{K} - \dim Z_{\mathbf{K}}(a_\lambda (a_\lambda)^{-\tau}) = 2(\dim \mathbf{A} - \dim Z_{\mathbf{A}}(a_\lambda)). \qquad (2.3.11)$$

But, by (1.2.4), we see that the right hand side is equal to

$$\{\dim \mathbf{G} - \dim Z_{\mathbf{G}}(a_\lambda (a_\lambda)^{-\tau})\}/2.$$

Hence (2.3.11) is equivalent to

$$\dim \mathbf{K} - \dim Z_{\mathbf{K}}(X) = \{\dim \mathbf{G} - \dim Z_{\mathbf{G}}(X)\}/2. \qquad (2.3.12)$$

By our calculation of $Z_{\mathbf{G}}(X)$ and $Z_{\mathbf{K}}(X)$, we see that both sides of this formula depend only on λ and do not depend on the characteristic p of k. Hence it is enough to prove this assuming $p \gg 0$ (or assuming $p = 0$). Let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of the algebraic group $\mathbf{G}$ (resp. $\mathbf{K}$). Then the automorphism τ of $\mathbf{G}$ induces an involutive automorphism of $\mathfrak{g}$, which will also be denoted by τ. We have $\mathfrak{k} = \mathfrak{g}_\tau$ and $\tau(X) = -X$. Hence, by a general result of Kostant and Rallis [10, p. 770, Prop. 5] (a proof can also be found in [8, (3.1.6)(vii)]), we get

$$\dim[\mathfrak{k}, X] = (\dim[\mathfrak{g}, X])/2,$$

from which (2.3.12) follows. This completes the proof of Proposition 2.3.6.

2.3.13. *Remark.* We denote by $\mathbf{P}$ the subset $\{xx^{-\tau}; x \in \mathbf{G}\}$ of $\mathbf{G}$, and by $V(\mathbf{P})$ the set of unipotent elements of $\mathbf{P}$ (see [14]). Let $V(\mathbf{A})$ be the set of unipotent elements of $\mathbf{A}$. Then, by 2.3.6, we have

$$|V(\mathbf{P})(\mathbf{F}_q)| = |V(\mathbf{A})(\mathbf{F}_q)|_{q \to q^2}.$$

Since we know (see, e.g., [17, III, 1.20]) that

$$|V(\mathbf{A})(\mathbf{F}_q)| = q^{\dim V(\mathbf{A})} = q^{n^2 - n},$$

we get

$$|V(\mathbf{P})(\mathbf{F}_q)| = q^{\dim V(\mathbf{P})} = q^{2(n^2 - n)}.$$

This answers the question posed in [8, p. 587] in the present case.

2.4. Let $a \in \mathrm{Cl}_A(a_\Lambda)$ $(\Lambda \in \mathscr{P}_{n,q})$ be a semisimple element of A. We consider the Hecke algebra

$$\mathscr{H}(a) = \mathscr{H}(Z_G(aa^{-\tau}), Z_K(aa^{-\tau})).$$

By (1.1.1) and (2.3.9), we get a natural isomorphism

$$\mathscr{H}(a) \simeq \bigotimes_{\alpha \in O(M)} \mathscr{H}(\mathrm{GL}_{2|\Lambda(\alpha)|}(\mathbf{F}_{q^{|\alpha|}}), \mathrm{Sp}_{2|\Lambda(\alpha)|}(\mathbf{F}_{q^{|\alpha|}})).$$

Hence, Lemma 2.3.4, Theorem 2.3.5, and Proposition 2.3.6 can easily be generalized to the present case (by replacing G, K, and A with $Z_G(aa^{-\tau})$, $Z_K(aa^{-\tau})$, and $Z_A(a)$, respectively).

2.4.1. LEMMA. *Let s be a semisimple element of A, and u a unipotent element of $Z_A(s)$. Then we have*

$$\mathscr{H}-\mathrm{ind}\, su = (\mathscr{H}-\mathrm{ind}\, s)(\mathscr{H}(s)-\mathrm{ind}\, u).$$

This follows from 2.2.1(ii) (see (2.3.7)).

3. BASIC FUNCTIONS ON THE HECKE ALGEBRA $\mathscr{H}(\mathrm{GL}_{2n}(\mathbf{F}_q), \mathrm{Sp}_{2n}(\mathbf{F}_q))$

The main purpose of this section is to define *basic functions* $\chi_T^{\mathscr{H}}[\theta]$ on the Hecke algebra $\mathscr{H} = \mathscr{H}(G, K)$ ($G = \mathrm{GL}_{2n}(\mathbf{F}_q)$, $K = \mathrm{Sp}_{2n}(\mathbf{F}_q)$), and prove some of their properties.

3.1. Let a be a semisimple element of the subgroup A of G defined in (2.3.3), and $\mathscr{I}$ the Hecke algebra $\mathscr{H}(a)$ defined in 2.4. We put

$$G(\mathscr{I}) = Z_G(aa^{-\tau}), \qquad K(\mathscr{I}) = Z_K(aa^{-\tau})$$

and

$$A(\mathscr{I}) = A \cap G(\mathscr{I}) = Z_A(a),$$

so that $\mathscr{I} = \mathscr{H}(G(\mathscr{I}), K(\mathscr{I}))$. The double cosets $K(\mathscr{I})\backslash G(\mathscr{I})/K(\mathscr{I})$ are in natural 1–1 correspondence with the conjugate classes of $A(\mathscr{I})$. Let R (resp. $R(\mathscr{I})$) be the subgroup of G (resp. $G(\mathscr{I})$) consisting of all the elements of the form

$$\begin{pmatrix} 1_n & l \\ 0 & 1_n \end{pmatrix},$$

where l is an upper-triangular nilpotent n-by-n matrix over $\mathbf{F}_q$. For $x, y \in A(\mathscr{I})$, we put

$$m_{x,y}^{\mathscr{I}} = |\{r \in R(\mathscr{I});\ yr(yr)^{-\tau} \in \mathrm{Cl}_{K(\mathscr{I})}(xx^{-\tau})\}|.$$

3.1.1. LEMMA. (i) *The number $m_{x,y}^{\mathscr{I}}$ depends only on the classes $\mathrm{Cl}_{A(\mathscr{I})}(x)$ and $\mathrm{Cl}_{A(\mathscr{I})}(y)$.*

(ii) *$m_{x,y}^{\mathscr{I}} = 0$ if the semisimple parts of x and y are not $A(\mathscr{I})$-conjugate.*

(iii) *Let s be a semisimple element of $A(\mathscr{I})$, and u, v unipotent elements of $Z(s) = Z_{A(\mathscr{I})}(s)$. Then*

$$m^{\mathscr{I}}_{su,sv} = |A(\mathscr{I})/(Z(s)|_q\, m^{\mathscr{I}(s)}_{u,v},$$

where $\mathscr{I}(s) = H(Z_{G(\mathscr{I})}(ss^{-\tau}), Z_{K(\mathscr{I})}(ss^{-\tau})) = H(as)$.

(iv) *Let x, y be elements of $A(\mathscr{I})$. Then $m^{\mathscr{I}}_{x,x} \neq 0$. Moreover $m^{\mathscr{I}}_{x,y} = 0$ unless y is contained in the Zariski-closure of* $\mathrm{Cl}_{\mathbf{A}(\mathscr{I})}(x)$ $(\mathbf{A}(\mathscr{I}) = Z_{\mathbf{A}}(a))$.

Proof. For simplicity of notations, we consider here only the case when $a = 1$ (hence $\mathscr{I} = \mathscr{H}$). But, as will be clear, the general case can be proved completely similarly.

(i) Let $y \in A$ and $r \in R$. Then $yr(yr)^{-\tau} = yy^{-\tau} + (rr^{-\tau} - 1_{2n})$. Moreover, for $b \in A$, we have

$$bb^{\tau}(yy^{-\tau}rr^{-\tau})(bb^{\tau})^{-1} = byb^{-1}(byb^{-1})^{-\tau}\,(bb^{\tau}r)(bb^{\tau}r)^{-\tau}.$$

Since $bb^{\tau}r(bb^{\tau}r)^{-\tau} = r'(r')^{-\tau}$ for some $r' \in R$, we see that, for a fixed x, $m^{\mathscr{H}}_{x,y}$ depends only on $\mathrm{Cl}_A(y)$. That, for a fixed y, $m^{\mathscr{H}}_{x,y}$ depends only on $\mathrm{Cl}_A(x)$ follows from 2.2.1(i) and 2.3.4.

(ii) Let s (resp. t) be the semisimple parts of x (resp. y). Then the semisimple parts of $xx^{-\tau}$ (resp. $yr(yr)^{-\tau}$ $(r \in R)$) is $ss^{-\tau}$ (resp. is conjugate to $tt^{-\tau}$). Part (ii) follows from this fact.

(iii) For a fixed semisimple element s of A, let $R_1 = Z_R(ss^{-\tau})$. Then there exists a subgroup R_2 of R such that R can be written as a direct product $R_1 \cdot R_2$. Let $r = r_1 r_2$ $(r_i \in R_i)$ be any element of R. Then $svr(svr)^{-\tau}$ is G-conjugate to $(ss^{-\tau})(vv^{-\tau})(r_1 r_1^{-\tau})$. Hence we have

$$m^{H}_{su,sv} = |R_2|\, m^{H(s)}_{u,v} = |A/Z_A(s)|_q\, m^{H(s)}_{u,v},$$

as required.

(iv) Suppose $m^{H}_{x,y} \neq 0$. Then there exists an element r of R such that $uu^{-\tau}$ is G-conjugate to $yr(yr)^{-\tau}$. Let X be the nilpotent matrix such that $yr(yr)^{-r} = yy^{-\tau} + X$. Then $yy^{-\tau} + X$ is $GL_{2n}(k)$-conjugate to $yy^{-\tau} + cX$ for any $c \in k^{\times}$. (Recall $k = \bar{\mathbf{F}}_q$.) Hence $yy^{-\tau}$ is in the closure of $\mathrm{Cl}_{\mathbf{G}}(xx^{-\tau})$.

This is equivalent to saying that y is in the closure of $\mathrm{Cl}_{\mathbf{A}}(x)$. This proves the second statement of (iv). The first one is trivial.

3.2. Let T be a maximal torus of A contained in $A(\mathscr{I})$, and θ a character of T. Let $\zeta^{A(\mathscr{I})}_T[\theta]$ be the virtual character of $A(\mathscr{I})$ defined in 1.2. We define the *basic function* $\chi^{\mathscr{I}}_T[\theta]$ on $\mathscr{I}$ by

$$\chi^{\mathscr{I}}_T[\theta]([x]) = \sum_{y \in A(\mathscr{I})} m^{\mathscr{I}}_{x,y}\, \omega(\zeta^{A(\mathscr{I})}_T[\theta])(y), \qquad x \in A, \tag{3.2.1}$$

where $[x] = \mathscr{I} - \operatorname{ind} x \cdot e(K(\mathscr{I}))\, xe(K(\mathscr{I})) \in \mathscr{I}$. Here we used the following notation: if ζ is a function on a group H and $\zeta(1) \neq 0$, then $\omega(\zeta)$ is the function on H defined by

$$\omega(\zeta)(h) = \zeta(h)\, \zeta(1)^{-1}, \qquad h \in H.$$

As we shall see, the role of basic functions $\chi_T^{\mathscr{I}}[\theta]$ in the character theory of the Hecke algebra $\mathscr{I}$ is quite analogous to that of $\zeta_T^{A(\mathscr{I})}[\theta]$'s in the character theory of the group $A(\mathscr{I})$. For example, the next result can be compared with (1.2.6).

3.2.2. LEMMA. *Let $\mathscr{I}$, T, $\theta, \ldots$ be as above. Let s be a semisimple element of $A(\mathscr{I})$, and u a unipotent element of $Z(s) = Z_{A(\mathscr{I})}(s) = A(\mathscr{I}(s))$. Then we have*

$$\chi_T^{\mathscr{I}}[\theta]([su]) = |A(\mathscr{I})/Z(s)|_q^2\, \varepsilon(A(\mathscr{I}))\, \varepsilon(Z(s))$$

$$\times \sum_{\substack{x \in A(\mathscr{I})/Z(s) \\ xsx^{-1} \in T}} \chi_{x^{-1}Tx}^{\mathscr{I}(s)}[1]([u])\, \theta(xsx^{-1}).$$

(*See* (1.2.7) *for the notation* $\varepsilon(\cdot)$.)

Proof. By 3.1.1(i)(ii), we have

$$\chi_T^{\mathscr{I}}[\theta]([su]) = |A(\mathscr{I})/Z(s)| \sum_{v \in V(Z(s))} m_{su,sv}^{\mathscr{I}}\, \omega(\zeta_T^{A(\mathscr{I})}[\theta])(sv),$$

where $V(Z(s))$ is the set of unipotent elements of $Z(s)$. By 3.1.1(ii)(iii), (1.2.6), and (1.2.7), the right hand side is equal to $|A(\mathscr{I})/Z(s)|_q^2\, |Z(s)|^{-1} \cdot \varepsilon(A(\mathscr{I}))\, \varepsilon(Z(s))$ times

$$\sum_{v \in Z(s)} m_{u,v}^{\mathscr{I}(s)} \sum_{\substack{x \in A \\ xsx^{-1} \in T}} \omega(\zeta_{x^{-1}Tx}^{Z(s)}[1])(v)\, \theta(xsx^{-1})$$

$$= \sum_{\substack{x \in A \\ xsx^{-1} \in T}} \chi_{x^{-1}Tx}^{\mathscr{I}(s)}[1]([u])\, \theta(xsx^{-1}),$$

as required.

3.3. Our next task is to show that the basic function $\chi_T^{\mathscr{I}}[\theta]$ is an irreducible character of the Hecke algebra $\mathscr{I}$ if θ is "in general position." For simplicity of notations, we formulate and prove the results only in the case $\mathscr{I} = \mathscr{H}$; the general case can be treated completely similarly.

Let ζ be an irreducible character of A. We define an element h_ζ of $\mathscr{H} = \mathscr{H}(G, K) = e \cdot \mathbf{C}G \cdot e$ by

$$h_\zeta = \sum_{A \in P_{n,q}} \left\{ \sum_{y \in A} m^{\mathscr{H}}_{a_A, y} \omega(\zeta)(y^{-1}) \right\} e a_A e. \tag{3.3.1}$$

(See 3.2 for the notation $\omega(\zeta)$.) Let Q be the parabolic subgroup of G consisting of the elements of the form

$$\begin{pmatrix} b & c \\ 0 & a \end{pmatrix},$$

where a, b, and c are n-by-n matrices over $\mathbf{F}_q$. We denote by $\zeta \otimes \zeta$ the irreducible character

$$\begin{pmatrix} b & c \\ 0 & a \end{pmatrix} \to \zeta \begin{pmatrix} 1_n & 0 \\ 0 & a \end{pmatrix} \zeta \begin{pmatrix} 1_n & 0 \\ 0 & b \end{pmatrix}$$

of Q.

3.3.2. Lemma. *If the induced characters $(\zeta \otimes \zeta)^G$ and $(1_K)^G$ have exactly m irreducible constituents $\eta_1, \eta_2, \ldots, \eta_m$ in common, then we have*

$$h_\zeta = \sum_{i=1}^{m} c_i \varepsilon_{\eta_i},$$

with some $c_i \in \mathbf{C}$. Here ε_{η_i} is the primitive idempotent of $\mathscr{H}$ corresponding to η_i in the sense of 2.1.

Proof. We consider the central primitive idempotent

$$f_\zeta = \zeta(1)^2 \, |Q|^{-1} \sum_{x \in Q} (\zeta \otimes \zeta)(x^{-1}) x$$

of $\mathbf{C}Q$ corresponding to $\zeta \otimes \zeta$. It is clearly enough to show that $e f_\zeta e$ is a constant-multiple of h_ζ. To see this, we first note that

$$yr \to yr(yr)^{-\tau}, \qquad y \in A, r \in R$$

is a bijective map from AR to the set $\{xx^{-\tau}; x \in Q\}$. Using this fact and 2.2.1(i), we see that $e f_\zeta e$ is a constant-multiple of

$$\sum_{\substack{y \in A \\ r \in R}} \left\{ \sum_{z \in Q_\tau} (\zeta \otimes \zeta)((yrz)^{-1}) \right\} eyre.$$

But, for a fixed $yr \in AR$,

$$\sum_{z \in Q_r} (\zeta \otimes \zeta)((yrz)^{-1}) = |R| \sum_{a \in A} \zeta({}^r a^{-1})\, \zeta(ay^{-1})$$
$$= |R|\; |A|\; \omega(\zeta)(y^{-1}).$$

Hence, $ef_\zeta e$ is a constant-multiple of

$$\sum_{\substack{y \in A \\ r \in R}} \omega(\zeta)(y^{-1})\, eyre,$$

which is equal to h_ζ by 2.2.1(i) and 2.3.4. This proves the lemma.

3.3.3. Lemma. *Let* Irr(A) *be the set of irreducible characters of A. For an element ζ of* Irr(A), *let h_ζ be the element of $\mathscr{H}$ defined by* (3.3.1).

(i) *Let ζ_1 and ζ_2 be elements of* Irr(A). *If the semisimple parts* (*see* 1.2.11) *of ζ_1 and ζ_2 are distinct, then $h_{\zeta_1} h_{\zeta_2} = 0$.*

(ii) *$\{h_\zeta;\, \zeta \in \mathrm{Irr}(A)\}$ is a basis of $\mathscr{H}$.*

(iii) *Let ζ_0 be a semisimple irreducible character* (*see* 1.2.11) *of A. Let ζ_1 be an element of*

$$\mathrm{Irr}(A; \zeta_0) = \{\zeta \in \mathrm{Irr}(A);\, \zeta_s = \zeta_0\},$$

where ζ_s is the semisimple part of ζ. Let η be a common irreducible constituent of $(\zeta \otimes \zeta)^G$ and $(1_K)^G$, and ε_η the idempotent of $\mathscr{H}$ corresponding to η. Then ε_η can be written as a linear combination of $\{h_\zeta;\, \zeta \in \mathrm{Irr}(A; \zeta_0)\}$.

(iv) *Let T be a maximal torus of A, and θ a character of T. Assume that $\pm\zeta_T^A[\theta]$* (*see* 1.2) *is an irreducible character of A. Then the* **C**-*valued function $\chi_T^{\mathscr{H}}[\theta]$ on $\mathscr{H}$ defined by* (3.2.1) *is an irreducible character of $\mathscr{H}$.*

Proof. (i) By our assumption and 1.2.13(i), the induced characters $(\zeta_i \otimes \zeta_i)^G$, $i = 1, 2$, have no irreducible constituent in common. Hence (i) follows from 3.3.2.

(ii) By 2.3.4, we have

$$\dim_{\mathbf{C}} \mathscr{H} = |\mathrm{Irr}(A)|.$$

Hence, it is enough to show the linear independence of $\{h_\zeta;\, \zeta \in \mathrm{Irr}(A)\}$. By (i), this is reduced to show the linear independence of

$$\{h_\zeta;\, \zeta \in \mathrm{Irr}(A; \zeta_0)\} \tag{3.3.4}$$

for any semisimple irreducible character ζ_0. (For the definition of

$\operatorname{Irr}(A; \zeta_0)$, see the statement of the part (iii).) Suppose this is not true. Then there exists a non-trivial linear relation

$$\sum_{\zeta \in \operatorname{Irr}(A;\zeta_0)} c_\zeta h_\zeta = 0$$

with $c_\zeta \in \mathbf{C}$. In particular, we have

$$\sum_{\zeta} c_\zeta \left\{ \sum_{v \in V(A)} m^{\mathscr{H}}_{u,v} \omega(\zeta)(v^{-1}) \right\} = 0, \qquad u \in V(A),$$

where $V(A)$ is the set of unipotent elements of A. Hence, using 3.1.1(iv) and the induction on the partial ordering (1.2.1) on the unipotent classes of A, we see

$$\sum_{\zeta \in \operatorname{Irr}(A;\zeta_0)} c_\zeta \omega(\zeta)(u^{-1}) = 0, \qquad u \in V(A).$$

But this cannot be true, because by (1.2.9) the restrictions of the elements of $\operatorname{Irr}(A; \zeta_0)$ to $V(A)$ are linearly independent. Hence (3.3.4) must be linearly independent. This proves part (ii).

(iii) This follows from (ii) and 3.3.2.

(iv) Since the irreducible character $\zeta = \pm \zeta^A_T[\theta]$ is semisimple, and $\operatorname{Irr}(A; \zeta) = \{\zeta\}$, we see from (iii) that h_ζ is a constant-multiple of a primitive idempotent of $\mathscr{H}$. Hence, by comparing (3.3.1) with (2.1.2), and by observing that $\chi^{\mathscr{H}}_T[\theta](e) = 1$, we get (iv).

3.4. For $\Psi \in \hat{\mathscr{P}}_{n,q}$, let $W(\Psi)$, φ_Ψ, and θ_Ψ be as in 1.2. Using these notations, we define a function $\tilde{\chi}_\Psi$ on $\mathscr{H} = \mathscr{H}(G, K)$ by

$$\tilde{\chi}_\Psi = |A|_{q'} \, |W(\Psi)|^{-1} \sum_{w \in W(\Psi)} \varphi_{\Psi'}(w) \, |T_w|^{-1} \, \chi^{\mathscr{H}}_{T_w}[\theta_\Psi | T_w], \quad (3.4.1)$$

where $\Psi' \in \hat{\mathscr{P}}_{n,q}$ is defined by $\Psi'(\alpha) = \Psi(\alpha)'$, $\alpha \in O(L)$.

3.4.2. Lemma. *Let* $\zeta = \pm \zeta_\Psi$ $(\Psi \in \hat{\mathscr{P}}_{n,q})$ *be the irreducible character of* G *defined in* 1.2, *and* h_ζ *the element of* $\mathscr{H}$ *defined by* 3.3.1. *Then*

$$h_\zeta = \zeta(1)^{-1} \sum_{\Lambda \in \mathscr{P}_{n,q}} \tilde{\chi}_\Psi([a_\Lambda^{-1}]) \, e a_\Lambda e.$$

Proof. By (1.2.9), we have, for $\Lambda \in \mathscr{P}_{n,q}$,

$$\sum_{y\in A} m^{\mathscr{H}}_{a_\Lambda, y}\, \omega(\zeta)(y^{-1})$$

$$=\zeta(1)^{-1} \sum_{y\in A} m^{\mathscr{H}}_{a_\Lambda, y} \Big\{ |W(\Psi)|^{-1}$$

$$\times \sum_{w\in W(\Psi)} (\varphi_\Psi \otimes \mathrm{sgn})(w)\, \zeta^A_{T_w}[\theta_\Psi \mid T_w](y^{-1}) \Big\}$$

$$=\zeta(1)^{-1}\, \tilde{\chi}_\Psi([a_\Lambda^{-1}]).$$

Hence, the lemma follows from the definition (3.3.1) of h_ζ.

3.4.3. LEMMA. *Let $\zeta = \pm\zeta_\Phi$ ($\Phi \in \hat{\mathscr{P}}_{n,q}$) be an irreducible character of A. We put $(\Phi) = \{\Psi \in \hat{\mathscr{P}}_{n,q};\ \Psi_s = \Phi_s\}$ (see 1.2.11). Let η be a common irreducible constituent of $(\zeta \otimes \zeta)^G$ and $(1_K)^G$. Then the corresponding character $\eta \mid \mathscr{H}$ of $\mathscr{H}$ can be written as*

$$\eta \mid \mathscr{H} = \sum_{\Psi \in (\Phi)} c(\eta, \Psi)\, \tilde{\chi}_\Psi \tag{3.4.4}$$

for some $c(\eta, \Psi) \in \mathbf{C}$.

Proof. This follows from 3.4.2, 3.3.3(iii), and (2.1.2).

Lemma 3.4.3 shows that in order to calculate the character table of the Hecke algebra $\mathscr{H} = \mathscr{H}(G, K)$ the following two problems have to be solved:

(1) Determine the values $\{\chi^{\mathscr{H}}_T[\theta]([a_\Lambda])\}_{\Lambda \in \mathscr{P}_{n,q}}$ of the basic functions.

(2) Determine the coefficients $c(\eta, \Psi)$ in (3.4.4). In Section 5, we show that the problem (1) has a nice solution in terms of Green polynomials, which are also fundamental in the character theory of finite general linear groups (see Section 1). In Section 6, we discuss problem (2), and show that thee exists a nice inductive algorithm to determine $c(\eta, \Psi)$'s. For both of these purpose, we need to know the irreducible decomposition of $(1_K)^G$. This is the topic treated in the next section.

4. DECOMPOSITION OF THE INDUCED CHARACTER $(1_K)^G$

4.1. Let $G = \mathrm{GL}_{2n}(\mathbf{F}_q)$ and $K = \mathrm{Sp}_{2n}(\mathbf{F}_q)$. By 2.3.5, we know that the induced character $(1_K)^G$ is multiplicity-free. The next theorem gives its irreducible decomposition.

4.1.1. THEOREM. *Let Ω: $O(L) \to P$ be an element of $\hat{\mathscr{P}}_{2n,q}$ (see 1.2 for* *this notation). Then the irreducible character* $\pm\zeta_\Omega$ *(see* 1.2.10) *of* G *is contained in* $(1_K)^G$ *if and only if* Ω *is even; i.e., all the parts of* $\Omega(\alpha)$ *are even for any* $\alpha \in O(L)$.

4.1.2. *Remark.* Our proof of 4.1.1 given below was partly inspired by a proof [15, Ex. 2.2] (due to G. D. James and J. Saxl) of the (well-known) "Weyl group version" of 4.1.1. In [15; Prop. 3.2], a part of 4.1.1 is proved by a method different from ours.

4.2. To prove 4.1.1, we first observe that the number of even elements of $\hat{\mathscr{P}}_{2n,q}$ is equal to the number of irreducible constituents of $(1_K)^G$ or, equivalently, to the dimension of the Hecke algebra $\mathscr{H} = \mathscr{H}(G, K)$. In fact, if

$$\Phi\colon O(L) \ni \alpha \to (\Phi(\alpha)_1, \Phi(\alpha)_2, \ldots) \in \mathscr{P}$$

is an element of $\hat{\mathscr{P}}_{n,q}$, we can define an element 2Φ of $\hat{\mathscr{P}}_{2n,q}$ by

$$(2\Phi)(\alpha) = (2\Phi(\alpha)_1, 2\Phi(\alpha)_2, \ldots), \qquad \alpha \in O(L).$$

Hence, the number in question is equal to $|\hat{\mathscr{P}}_{n,q}|$, hence to $|\mathscr{P}_{n,q}|$, which is equal to $\dim \mathscr{H}$ by 2.3.4. Thus, for a proof of 4.1.1, it is enough to show:

4.2.1. Let $\Omega \in \hat{\mathscr{P}}_{2n,q}$. The irreducible character $\pm\zeta_\Omega$ of G is not contained in $(1_K)^G$ if Ω is not even.

4.3. For a proof of 4.2.1, we prepare three lemmas. To state the first one, we put

$$G_1 = \left\{ \begin{pmatrix} & & 0 \\ & X & \vdots \\ & & 0 \\ 0 & \cdots 0 & 1 \end{pmatrix}; X \in \mathrm{GL}_{2n-1}(\mathbf{F}_q) \right\} \subset G,$$

$$G_2 = G_1 \cap G_1^\tau \simeq \mathrm{GL}_{2n-2}(\mathbf{F}_q),$$

and

$$K_2 = (G_2)_\tau = G_1 \cap K \simeq \mathrm{Sp}_{2n-2}(\mathbf{F}_q).$$

Then it is easy to see that

$$G = G_1 K.$$

Hence, by Mackey's theorem, we have

4.3.1. Lemma. $(1_K)^G \mid G_1 = ((1_{K_2})^{G_2})^{G_1}$.

We also need the following two lemmas.

4.3.2. LEMMA (Thoma [18]). *Let θ (resp. λ) be the irreducible character of* $\mathrm{GL}_m(\mathbf{F}_q)$ *(resp.* $\mathrm{GL}_{m-1}(\mathbf{F}_q)$*) corresponding to* $\Theta \in \hat{\mathscr{P}}_{m,q}$ *(resp.* $\Lambda \in \hat{\mathscr{P}}_{m \times 1,q}$*). Then the multiplicity* $[\theta \mid \mathrm{GL}_{m-1}(\mathbf{F}_q) : \lambda] = [\lambda^{\mathrm{GL}_m(\mathbf{F}_q)} : \theta]$ *is equal to*

$$\prod_{\alpha \in O(L)} f(\Theta(\alpha), \Lambda(\alpha)),$$

where, for $\gamma = (\gamma_1, \gamma_2, ...)$ *and* $\delta = (\delta_1, \delta_2, ...) \in \mathscr{P}$,

$$f(\gamma, \delta) = \begin{cases} \prod_{d=1}^{\infty} (l_d + 1) & \text{if } \ |\gamma_i - \delta_i| \leqslant 1 \text{ for all } i, \\ 0 & \text{if } \ |\gamma_i - \delta_i| \geqslant 2 \text{ for some } i. \end{cases}$$

Here l_d *is the number of indices* i *such that* $\gamma_i = \delta_i = d$.

4.3.3. LEMMA. *Let* $\Omega \in \hat{\mathscr{P}}_{2n,q}$. *If the irreducible character* $\pm \zeta_\Omega$ *of* G *is contained in* $(1_K)^G$, *then* $|\Omega(\alpha)|$ *is even for any* $\alpha \in O(L)$.

Proof. By 3.3.2 and 3.3.3(ii), we see that any irreducible constituent of $(1_K)^G$ is contained in $(\zeta \otimes \zeta)^G$ for some irreducible character ζ of A. Hence the lemma follows from 1.2.13(i).

4.4. *Proof of* 4.2.1. We prove this by induction on n. When $n = 0$, i.e., when $G = K = \{1\}$, this is trivial. By the induction hypothesis, we have:

4.4.1. An irreducible character ζ of G_2 is contained in $(1_{K_2})^{G_2}$ if and only if $\zeta = \pm \zeta_\Lambda$ for some even $\Lambda \in \hat{\mathscr{P}}_{2n-2,q}$.

4.4.2. We assume: there exists an element Ω of $\hat{\mathscr{P}}_{2n,q}$ such that Ω is not even, and that the corresponding irreducible character $\pm \zeta_\Omega$ of G is contained in $(1_K)^G$. We define an even integer $l(\Omega) \geqslant 2$ by

$$l(\Omega) = \sum_{\alpha \in O(L)} |\alpha| \, |\{i; \Omega(\alpha)_i \in 2\mathbf{Z} + 1\}|,$$

where we write $\Omega(\alpha) = (\Omega(\alpha)_1, \Omega(\alpha)_2, ...)$. We fix an element β_0 of $O(L)$ such that $|\beta_0| = l(\Omega) - 1$. We define $\Xi^* \in \hat{\mathscr{P}}_{2n - l(\Omega),q}$ and $\Xi \in \hat{\mathscr{P}}_{2n-1,q}$ by

$$\Xi^*(\alpha)_i = \begin{cases} \Omega(\alpha) & \text{if } \ \Omega(\alpha)_i \in 2\mathbf{Z}, \\ \Omega(\alpha)_i - 1 & \text{otherwise,} \end{cases}$$

and

$$\Xi(\alpha)_i = \begin{cases} \Xi^*(\alpha)_i + 1 & \text{if } \ (\alpha, i) = (\beta_0, 1) \\ \Xi^*(\alpha)_i & \text{otherwise,} \end{cases}$$

for $\alpha \in O(L)$ and $i = 1, 2, 3, \ldots$. By 4.3.2, the irreducible character $\pm\zeta_{\Xi}$ of $G_1 \simeq \mathrm{GL}_{2n-1}(\mathbf{F}_q)$ is contained in $\pm\zeta_{\Omega}|G_1$. Hence, by 4.3.1 and 4.4.1, it must be contained in $(\pm\zeta_{\Lambda})^{G_1}$ for some even $\Lambda \in \hat{\mathscr{P}}_{2n-2,q}$. Then, by 4.3.2, $|\Lambda(\alpha)_i - \Xi(\alpha)_i| \leqslant 1$ for any (α, i). Since $\Lambda(\alpha)_i$ is even for any (α, i), and $\Xi(\alpha)_i$ is even if and only if $(\alpha, i) \neq (\beta_0, 1)$, such $\Lambda \in \hat{\mathscr{P}}_{2n-2,q}$ must be equal to $\Xi^* \in \hat{\mathscr{P}}_{2n-l(\Omega),q}$ defined above. Hence we have $l(\Omega) = 2$. By this equality and 4.3.3, we see that there exists an element α_0 of $O(L)$, and positive integers i_0, j_0 such that

(a) $|\alpha_0| = 1$, $i_0 < j_0$,

(b) $\Omega(\alpha)_i \in 2\mathbf{Z}$ unless $(\alpha, i) = (\alpha_0, i_0)$ or (α_0, j_0), and

(c) $\Omega(\alpha_0)_{i_0}, \Omega(\alpha_0)_{j_0} \in 2\mathbf{Z} + 1$.

4.4.3. First we consider the case when $\Omega(\alpha_0)_{j_0} = 1$ or, equivalently, the case when $\Omega(\alpha)_i = 1$ for some (α, i). Let Π be an element of $\hat{\mathscr{P}}_{2n-1,q}$ defined by

$$\Pi(\alpha)_i = \begin{cases} 0 & \text{if} \quad (\alpha, i) = (\alpha_0, j_0), \\ \Omega(\alpha)_i & \text{otherwise.} \end{cases}$$

Then, by 4.3.2, the irreducible character $\pm\zeta_{\Pi}$ of G_1 is contained in $\pm\zeta_{\Omega}|G_1$. Hence, by 4.3.1 and 4.4.1, it must be contained in $(\pm\zeta_{\Lambda})^{G_1}$ for some even $\Lambda \in \hat{\mathscr{P}}_{2n-2,q}$. By 4.3.2, such a Λ is unique and given by

$$\Lambda(\alpha)_i = \begin{cases} \Omega(\alpha)_i - 1 & \text{if} \quad (\alpha, i) = (\alpha_0, i_0) \text{ or } (\alpha_0, j_0), \\ \Omega(\alpha)_i & \text{otherwise.} \end{cases}$$

Hence, by 4.3.1, we must have

$$[\pm\zeta_{\Omega}|G_1 : \pm\zeta_{\Pi}] \leqslant [(\pm\zeta_{\Lambda})^{G_1} : \pm\zeta_{\Pi}].$$

But, by 4.3.2 and the definitions of Λ and Π, we have

$$[\pm\zeta_{\Omega}|G_1 : \pm\zeta_{\Pi}] = 2[(\pm\zeta_{\Lambda})^{G_1} : \pm\zeta_{\Pi}],$$

which contradicts to the above inequality. Hence $\Omega(\alpha)_i \neq 1$ for any pair (α, i).

4.4.4. Let $m(\Omega)$ be an integer $\geqslant 2$ defined by

$$m(\Omega) = \sum_{\substack{\alpha \in O(L) \\ |\alpha| = 1}} |\{i; \Omega(\alpha)_i \neq 0\}|.$$

Let γ_0 be an element of $O(L)$ such that $|\gamma_0| = m(\Omega) - 1$. Let $t_0 \geqslant 1$ be the minimum number such that $\Omega(\gamma_0)_{t_0} = 0$. We define $\Phi \in \hat{P}_{2n-1,q}$ by

$$\Phi(\alpha)_i = \begin{cases} \Omega(\alpha)_i - 1 & \text{if } \ |\alpha| = 1 \text{ and } \Omega(\alpha)_i \neq 0, \\ 1 & \text{if } \ (\alpha, i) = (\gamma_0, t_0), \\ \Omega(\alpha)_i & \text{otherwise.} \end{cases}$$

Then $[\pm\zeta_\Omega | G_1 : \pm\zeta_\Phi] \neq 0$. Hence $[(\pm\zeta_\Lambda)^{G_1} : \pm\zeta_\Phi] \neq 0$ for some even $\Lambda \in \hat{\mathscr{P}}_{2n-2,q}$. By 4.3.2, such Λ is unique and given by

$$\Lambda(\alpha)_i = \begin{cases} \Omega(\alpha)_i - 1 & \text{if } \ (\alpha, i) = (\alpha_0, i_0) \text{ or } (\alpha_0, j_0), \\ \Omega(\alpha)_i & \text{otherwise.} \end{cases}$$

Note that, by 4.4.3, $\Phi(\alpha_0)_{i_0}$ $(= \Omega(\alpha_0)_{i_0} - 1) \neq 0$ and $\Phi(\alpha_0)_{j_0}$ $(= \Omega(\alpha_0)_{j_0} - 1)$ $\neq 0$. Hence, by 4.3.2, we have

$$[(\pm\zeta_\Lambda)^{G_1} : \pm\zeta_\Phi] = e[\pm\zeta_\Omega | G_1 : \pm\zeta_\Phi],$$

where $e = 3$ or 4 according to whether $\Omega(\alpha_0)_{i_0} = \Omega(\alpha_0)_{j_0}$ or $\Omega(\alpha_0)_{i_0} \neq \Omega(\alpha_0)_{j_0}$. Hence there exists an $\Omega^* \in \hat{\mathscr{P}}_{2n,q}$ different from Ω such that $\pm\zeta_{\Omega^*}$ is contained in $(1_K)^G$ and that

$$0 \neq [\pm\zeta_{\Omega^*} | G_1 : \pm\zeta_\Phi] \leqslant (e-1)[\pm\zeta_\Omega | G_1 : \pm\zeta_\Phi]. \qquad (\#)$$

If Ω^* is even, we see that $\Omega^*(\alpha)_i = \Phi(\alpha)_i$ if and only if $\Lambda(\alpha)_i = \Phi(\alpha)_i$. This means that

$$[\pm\zeta_{\Omega^*} | G_1 : \pm\zeta_\Phi] = [(\pm\zeta_\Lambda)^{G_1} : \pm\zeta_\Phi] = e[\pm\zeta_\Omega | G_1 : \pm\zeta_\Phi],$$

which contradicts to the second inequality in $(\#)$. Hence Ω^* is not even. Thus Ω^* satisfies the same conditions which were imposed on Ω at the beginning of 4.4.2.

4.4.5. We claim: if $\Omega(\alpha)_i = 0$ and $(\alpha, i) \neq (\gamma_0, t_0)$, then $\Omega^*(\alpha)_i = 0$. In fact, if $\Omega(\alpha)_i = 0$ and $(\alpha, i) \neq (\gamma_0, t_0)$, then $\Phi(\alpha)_i = 0$. Hence, by 4.3.2, we have $\Omega^*(\alpha)_i = 0$ or 1. But, by 4.4.4 and 4.4.3, $\Omega^*(\alpha)_i \neq 1$ for any (α, i). Hence we get the claim.

4.4.6. By 4.4.2 and 4.4.4, the partitions $\Omega(\alpha)$ and $\Omega^*(\alpha)$ are even if $|\alpha| \neq 1$. Hence, by 4.3.2, $\Omega^*(\alpha)_i = \Phi(\alpha)_i = \Omega(\alpha)_i$ if $|\alpha| \neq 1$ and $(\alpha, i) \neq (\gamma_0, t_0)$. Hence we have

$$\sum_{|\alpha|=1} \Omega(\alpha)_i = \Omega^*(\gamma_0)_{t_0} + \sum_{\substack{|\alpha|=1 \\ (\alpha, i) \neq (\gamma_0, t_0)}} \Omega^*(\alpha)_i.$$

4.4.7. By 4.3.2, $|\Omega^*(\alpha)_i - \Phi(\alpha)_i| \leqslant 1$ for any (α, i). Hence if $|\alpha| = 1$ and $\Omega(\alpha)_i \neq 0$, we have

$$0 \leqslant \Omega(\alpha)_i - \Omega^*(\alpha)_i \leqslant 1.$$

4.4.8. If $\Omega^*(\gamma_0)_{t_0} = 0$, then 4.4.5, 4.4.6, and 4.4.7 imply that Ω and Ω^* must coincide, contrary to our assumption. Hence $\Omega^*(\gamma_0)_{t_0} \neq 0$.

4.4.9. Assume $|\gamma_0| = m(\Omega) - 1 = 1$. Then, by 4.4.5, $\Omega^*(\alpha)_i = 0$, if $|\alpha| = 1$ and $(\alpha, i) \neq (\alpha_0, i_0), (\alpha_0, j_0), (\gamma_0, t_0)$. Moreover, we have $\Omega^*(\alpha_0)_{i_0} \neq 0$ and $\Omega^*(\alpha_0)_{j_0} \neq 0$ by 4.4.3 and 4.4.7. Hence, by 4.4.8, we have

$$m(\Omega^*) = 3.$$

We also have

$$\sum_{|\alpha| = 1} \Omega(\alpha)_i = \sum_{|\alpha| = 1} \Omega^*(\alpha)_i$$

by 4.4.6.

4.4.10. If $|\gamma_0| = m(\Omega) - 1 \geqslant 2$, we get, from 4.4.6 and 4.4.8,

$$\sum_{|\alpha| = 1} \Omega(\alpha)_i > \sum_{|\alpha| = 1} \Omega^*(\alpha)_i.$$

4.4.11. We now finish our proof of 4.2.1. Let Ω be as at the beginning of 4.4.2. We put

$$N(\Omega) = \sum_{|\alpha| = 1} \Omega(\alpha)_i,$$

which is equal to or greater than 2 by 4.4.2. We prove the non-existence of Ω by induction on $N(\Omega)$. If $N(\Omega) = 2$, this follows from 4.4.3. The general case follows from the induction assumption, and 4.4.4, 4.4.9, 4.4.10. This completes the proof of 4.2.1, and hence, that of 4.1.1.

5. Values of Basic Functions

The basic functions $\chi_T^{\mathscr{H}}[\theta]$ on the Hecke algebra $\mathscr{H}$ were introduced in Section 3. Here we prove a simple formula (Theorem 5.3.2) expressing the values of $\chi_T^{\mathscr{H}}[\theta]$ in terms of Green's virtual character $\zeta_T^A[\theta]$ of $A \simeq \mathrm{GL}_n(\mathbf{F}_q)$.

5.1. Let $\mathbf{B}$ be the subgroup of $\mathbf{G} = \mathrm{GL}_{2n}(k)$ ($k = \bar{\mathbf{F}}_q$) consisting of the elements of $\mathbf{G}$ of the form

$$\begin{pmatrix} b_2 & m \\ 0 & b_1 \end{pmatrix},$$

where b_1, b_2, and m are n-by-n matrices over k and b_1 and b_2 are upper and lower triangular, respectively. Let $\mathscr{B}_\tau$ be the variety of τ-stable Borel subgroups of $\mathbf{G}$, i.e., $\mathscr{B}_\tau = \{x\mathbf{B}x^{-1}; x \in \mathbf{K}\}$. Let v be a unipotent element of $\mathbf{G}$ which can be written as

$$v = uu^{-\tau}$$

for some $u \in \mathbf{G}$. The following results are versions of results due to R. Steinberg and N. Spaltenstein (see [4, 5.10]).

5.1.1. LEMMA. (i) *Let*

$$\mathscr{B}_{\tau,v} = \{\mathbf{B}' \in \mathscr{B}_\tau; v \in \mathbf{B}'\}.$$

Then

$$\dim Z_{\mathbf{K}}(v) = \operatorname{rank} \mathbf{K} + 2 \dim \mathscr{B}_{\tau,v}.$$

(ii) $2 \dim \mathrm{Cl}_{\mathbf{K}}(v) \cap B = \dim \mathrm{Cl}_{\mathbf{K}}(v)$.

Proof. Let S be the closed subset of $\mathscr{B}_\tau \times \mathscr{B}_\tau \times \mathrm{Cl}_{\mathbf{K}}(v)$ defined by

$$S = \{(\mathbf{B}_1, \mathbf{B}_2, y) \in \mathscr{B}_\tau \times \mathscr{B}_\tau \times \mathrm{Cl}_{\mathbf{K}}(v);\ y \in \mathbf{B}_1 \cap \mathbf{B}_2\}.$$

Let $\mathbf{D}$ be the diagonal subgroup of $\mathbf{G}$. For each $w \in (N_{\mathbf{G}}(\mathbf{D})/\mathbf{D})_\tau$, we define S_w by

$$S_w = \{(\mathbf{B}_1, \mathbf{B}_2, y) \in S; \mathbf{B}_1 = x\mathbf{B}x^{-1}, \mathbf{B}_2 = xw\mathbf{B}w^{-1}x^{-1} \text{ for some } x \in \mathbf{K}\}.$$

Then S is the disjoint union of S_w. Hence we have

$$\dim S \geqslant \dim S_w$$

with equality for some w. On the other hand, we have

$$\dim S = \dim \mathrm{Cl}_{\mathbf{K}}(v) + \dim(\mathscr{B}_{\tau,v} \times \mathscr{B}_{\tau,v})$$

and

$$\dim S_w = \dim \mathbf{K}/(\mathbf{B} \cap w\mathbf{B}w^{-1})_\tau + \dim(\mathrm{Cl}_{\mathbf{K}}(v) \cap \mathbf{B} \cap w\mathbf{B}w^{-1})_\tau.$$

Hence

$$\dim Z_{\mathbf{K}}(v) \leqslant \dim \mathbf{D}_\tau + 2 \dim \mathscr{B}_{\tau,v} + \dim(\mathbf{U}_w)_\tau - \dim \mathrm{Cl}_{\mathbf{K}}(v) \cap (\mathbf{U}_w)_\tau$$

with equality for some w. Here we put

$$\mathbf{U}_w = \mathbf{U} \cap w\mathbf{U}w^{-1},$$

where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$. Hence (i) will follow if we show

$$\mathrm{Cl}_{\mathbf{K}}(u) \cap \mathbf{U}_w \text{ is dense in } \mathbf{U}_w \tag{*}$$

for some w. But, by 2.2.1(i) and 2.3.4(ii), we can assume that $v = uu^{-\tau}$ for an upper unitriangular element u of $\mathbf{A}$. Then, if $\mathbf{U}'$ is the unipotent radical of $\mathbf{B} \cap \mathbf{A}$, $\mathrm{Cl}_{\mathbf{A}}(u) \cap \mathbf{U}' \cap w'\mathbf{U}'w'^{-1}$ is dense in $\mathbf{U}' \cap w'\mathbf{U}'w'^{-1}$ for some $w' \in N_{\mathbf{A}}(\mathbf{D})/(\mathbf{A} \cap \mathbf{D}) \subset N_{\mathbf{G}}(\mathbf{D})/\mathbf{D}$. Hence (*) holds if one puts

$$w = (w'w_0')^\tau\, w'w_0'w_0,$$

where w_0 is the element of $N_{\mathbf{G}}(\mathbf{D})/\mathbf{D}$ such that $\mathbf{U}_{w_0} = \{1\}$, and w_0' is the element of $N_{\mathbf{A}}(\mathbf{D})/(\mathbf{A} \cap \mathbf{D})$ such that $\mathbf{U}' \cap w_0'\mathbf{U}'w_0'^{-1} = \{1\}$. This proves (i).

(ii) We consider the following morphisms:

$$\pi_1 : \mathbf{K} \to \mathrm{Cl}_{\mathbf{K}}(v), \quad x \to x^{-1}vx,$$

$$\pi_2 : \mathbf{K} \to \mathscr{B}_\tau, \qquad x \to x\mathbf{B}x^{-1}.$$

Then $\pi_1^{-1}(\mathrm{Cl}_{\mathbf{K}}(v) \cap \mathbf{U}) = \pi_2^{-1}(\mathscr{B}_{\tau,v})$. We denote this variety by Y. We have

$$\dim Y = \dim \pi_1^{-1}(\mathrm{Cl}_{\mathbf{K}}(v) \cap \mathbf{U}) = \dim \mathrm{Cl}_{\mathbf{K}}(v) \cap \mathbf{U} + \dim Z_{\mathbf{K}}(v)$$

and

$$\dim Y = \dim \pi_2^{-1}(\mathscr{B}_{\tau,v}) = \dim \mathscr{B}_{\tau,v} + \dim \mathbf{B} \cap \mathbf{K}.$$

Hence

$$\dim(\mathrm{Cl}_{\mathbf{K}}(v) \cap \mathbf{U}) + \dim Z_{\mathbf{K}}(v) = \dim \mathscr{B}_{\tau,v} + \dim \mathbf{B} \cap \mathbf{K}.$$

Combining this with (i), we get (ii).

5.1.2. LEMMA. *Let* $\mathbf{R}$ *be the subgroup of* $\mathbf{G}$ *which consists of the elements of the form*

$$\begin{pmatrix} 1_n & l \\ 0 & 1_n \end{pmatrix},$$

where l is an upper triangular nilpotent n-by-n matrix over k. Let $\mathbf{S}$ *be the subgroup of* $\mathbf{K}$ *which consists of the elements of the form*

$$\begin{pmatrix} 1_n & m \\ 0 & 1_n \end{pmatrix},$$

where m is a symmetric n-by-n matrix over k. Let u be a unipotent element of $\mathbf{A}$. *Then we have*

$$\begin{aligned}\{x \in \mathrm{Cl}_{\mathbf{K}}(uu^{-\tau});\, x = ur(ur)^{-\tau} \text{ for some } r \in \mathbf{R}\}\\ = \{suu^{-\tau}s^{-1};\, s \in \mathbf{S}\}.\end{aligned} \tag{5.1.3}$$

Moreover, as a variety, this is the affine space of dimension $\dim \mathrm{Cl}_{\mathbf{A}}(u)/2$.

Proof. By matrix calculations, we see that the right hand side of (5.1.3) is isomorphic to

$$\{mu - {}^t um;\, m = {}^t m\},$$

which is an affine space. The dimension d_r of this affine space depends only on the element of $\mathscr{P}_n$ corresponding to the $\mathbf{A}$-conjugate class of u, and does not depend on the characteristic p of k. Assuming p is large, we have

$$d_r = \dim[\mathrm{Lie}\,\mathbf{S}, X] = \dim[\mathrm{Lie}\,{}^t\mathbf{S}, X],$$

where $X = uu^{-\tau} - 1_{2n} \in \mathrm{Lie}\,\mathbf{G}$ and ${}^t\mathbf{S} = \{{}^t s;\, s \in \mathbf{S}\}$. We also have

$$\dim[\mathrm{Lie}\,\tilde{\mathbf{A}}, X] = \dim \mathrm{Cl}_{\tilde{\mathbf{A}}}(uu^{-\tau}) = \dim \mathrm{Cl}_{\mathbf{A}}(u),$$

where $\tilde{\mathbf{A}} = \{aa^{\tau};\, a \in \mathbf{A}\} \subset \mathbf{K}$. Moreover, by (2.3.1),

$$[\mathrm{Lie}\,\mathbf{K}, X] = 2\dim \mathrm{Cl}_{\mathbf{A}}(u).$$

Hence, by using

$$[\mathrm{Lie}\,\mathbf{K}, X] = [\mathrm{Lie}\,\tilde{\mathbf{A}}, X] \oplus [\mathrm{Lie}\,\mathbf{S}, X] \oplus [\mathrm{Lie}\,{}^t\mathbf{S}, X],$$

we get

$$d_r = \dim \mathrm{Cl}_{\mathbf{A}}(u)/2.$$

Thus we have shown that the right hand side of (5.1.3) is the affine space of the required dimension. Next we prove the equality (5.1.3). The right hand side is clearly contained in the left hand side. Assume that there exists an element $ur_0(ur_0)^{-\tau}$ of the left hand side which is not contained in the right hand side. Then, since

$$a_c s(ur_0(ur_0)^{-\tau})s^{-1}a_c^{-1} = suu^{-\tau}s^{-1} + c^{-2}(r_0 r_0^{-\tau} - 1_{2n})$$

for $s \in \mathbf{S}$ and

$$a_c = \begin{pmatrix} c^{-1} 1_{2n} & 0 \\ 0 & c 1_{2n} \end{pmatrix} \qquad (c \in k^\times),$$

we see that

$$d_l > d_r, \tag{5.1.4}$$

where d_l is the dimension of the left hand side of (5.1.3). But, by 5.1.1(ii) and (the proof of) 3.1.1(i),

$$d_l \leqslant \dim \mathrm{Cl}_{\mathbf{K}}(uu^{-\tau})/2 - \dim \mathrm{Cl}_{\mathbf{A}}(u) \cap \mathbf{A} \cap \mathbf{B}.$$

Since we know

$$\dim \mathrm{Cl}_{\mathbf{A}}(u) \cap \mathbf{A} \cap \mathbf{B} = \dim \mathrm{Cl}_{\mathbf{A}}(u)/2$$

(this is due to Spaltenstein, see [4, 5.10.2]) and

$$\dim \mathrm{Cl}_{\mathbf{K}}(uu^{-\tau}) = 2 \dim \mathrm{Cl}_{\mathbf{A}}(u)$$

(see 2.3.6), we have

$$d_l \leqslant \dim \mathrm{Cl}_{\mathbf{A}}(u)/2.$$

Hence (5.1.4) cannot be true. This proves (5.1.3).

5.2. Let $\mathscr{I}$ be, as in Section 3, the Hecke algebra $\mathscr{H}(a) = \mathscr{H}(Z_G(aa^{-\tau}), Z_K(aa^{-\tau}))$ for a fixed semisimple element a of A. Let $A(\mathscr{I})$ and $m^{\mathscr{I}}_{x,y}$ $(x, y \in A(\mathscr{I}))$ be as in 3.1.

5.2.1. Lemma. *Let u be a unipotent element of $A(\mathscr{I})$. Then*

$$m^{\mathscr{I}}_{u,u} = q^{e(u)},$$

where $e(u) = \dim \mathrm{Cl}_{A(\mathscr{I})}(u)/2$.

Proof. When $\mathscr{I} = \mathscr{H}$, this follows from 5.1.2. The general case can be reduced to this special case (see 2.4).

5.3. Let $V(A(\mathscr{I}))$ be the set of unipotent elements of $A(\mathscr{I})$. For a maximal torus T of $A(\mathscr{I})$, we define a function $X^{\mathscr{I}}_T$ on $V(A(\mathscr{I}))$ by

$$X^{\mathscr{I}}_T(u) = \chi^{\mathscr{I}}_T[\theta]([u]), \qquad u \in V(A(\mathscr{I})),$$

where $\chi^{\mathscr{I}}_T[\theta]$ is the basic function (see Section 3) on $\mathscr{I}$ associated with T and a character θ of T. Note that, by 3.2.2, the right hand side does not depend on θ.

5.3.1. LEMMA. *For a maximal torus T of $A(\mathscr{I})$, let $\tilde{Q}_T^{A(\mathscr{I})}$ be the function on $V(A(\mathscr{I}))$ defined by* (1.3.1). *Then we have*

$$X_T^{A(\mathscr{I})}(u) = \{\tilde{Q}_T^{A(\mathscr{I})}(u)\}_{q \to q^2}, \qquad u \in V(A(\mathscr{I})).$$

(*See the paragraph just before Proposition* 2.3.6 *for the notation* $\{\ \cdots\ \}_{q \to q^2}$. *See also* 1.3.2(vi).)

A proof of this lemma will be given in 5.4.

Let $\zeta_T^{A(\mathscr{I})}[\theta]$ be the virtual character of $A(\mathscr{I})$ defined in 1.2. Let s and u be the semisimple and unipotent parts of an element x of $A(\mathscr{I})$. We put

$$\{\zeta_T^{A(\mathscr{I})}[\theta](x)\}_{q \to q^2} = \sum_{\substack{x \in A(\mathscr{I})/Z(s) \\ xsx^{-1} \in T}} \{Q_{x^{-1}Tx}^{Z(s)}(u)\}_{q \to q^2} \cdot \theta(xsx^{-1}),$$

where $Z(s) = Z_{A(\mathscr{I})}(s)$. The notation is justified by (1.2.6). We can now state the main result of this section.

5.3.2. THEOREM. *Let θ be a character of a maximal torus T of $A(\mathscr{I})$. Then, for any $x \in A(\mathscr{I})$, we have*

$$\chi_T^{A(\mathscr{I})}[\theta]([x]) = \{|\mathrm{Cl}_{A(\mathscr{I})}(x)|\ \zeta_T^{A(\mathscr{I})}[\theta](x)\ \zeta_T^{A(\mathscr{I})}[\theta](1)^{-1}\}_{q \to q^2}.$$

Proof. By 3.2.2 and 5.3.1, we see that $\chi_T^{A(\mathscr{I})}[\theta]([x])$ is equal to $\varepsilon(A(\mathscr{I}))\,\varepsilon(Z(s))$ times

$$\begin{aligned}
|A(\mathscr{I})/Z(s)|_q^2 & \sum_{\substack{x \in A(\mathscr{I})/Z(s) \\ xsx^{-1} \in T}} \{\tilde{Q}_{x^{-1}Tx}^{Z(s)}(u)\}_{q \to q^2} \cdot \theta(xsx^{-1}) \\
&= \sum_{\substack{x \in A(\mathscr{I})/Z(s) \\ xsx^{-1} \in T}} \{|A(\mathscr{I})/Z(s)|_q\ |\mathrm{Cl}_{Z(s)}(u)|\ Q_{x^{-1}Tx}^{Z(s)}(u) \\
&\qquad \cdot Q_{x^{-1}Tx}^{Z(s)}(1)^{-1}\}_{q \to q^2} \cdot \theta(xsx^{-1}) \\
&= \{|A(\mathscr{I})/Z(s)|_q\ |\mathrm{Cl}_{Z(s)}(u)|\ Q_{x^{-1}Tx}^{Z(s)}(1)^{-1}\}_{q \to q^2} \cdot \{\zeta_T^{A(\mathscr{I})}[\theta](x)\}_{q \to q^2}.
\end{aligned}$$

But, since $|\mathrm{Cl}_{A(\mathscr{I})}(x)| = |A(\mathscr{I})/Z(s)|\ |\mathrm{Cl}_{Z(s)}(u)|$, and since $\zeta_T^{A(\mathscr{I})}[\theta](1) = \varepsilon(A(\mathscr{I}))\,\varepsilon(Z(s))\ |A(\mathscr{I})/Z(s)|_{q'}\ Q_{x^{-1}Tx}^{Z(s)}(1)$ (see (1.2.7)), we have

$$\begin{aligned}
&|A(\mathscr{I})/Z(s)|_q\ |\mathrm{Cl}_{Z(s)}(u)|\ Q_{x^{-1}Tx}^{Z(s)}(1)^{-1} \\
&\qquad = \varepsilon(A(\mathscr{I}))\,\varepsilon(Z(s))\ |\mathrm{Cl}_{A(\mathscr{I})}(x)|\ \zeta_T^{A(\mathscr{I})}[\theta](1)^{-1}.
\end{aligned}$$

Hence we get

$$\chi_T^{\mathscr{I}}[\theta]([x]) = \{|\mathrm{Cl}_{A(\mathscr{I})}(x)|\ \zeta_T^{A(\mathscr{I})}[\theta](1)^{-1}\}_{q \to q^2} \cdot \{\zeta_T^{A(\mathscr{I})}[\theta](x)\}_{q \to q^2},$$

as required.

5.4. *Proof of* 5.3.1. By 1.3.2(v), it is enough to prove (I)–(IV) below:

(I) $X_T^{\mathscr{I}}(u) = X_T^{\mathscr{I}}(u')$ if $\mathrm{Cl}_{A(\mathscr{I})}(u) = \mathrm{Cl}_{A(\mathscr{I})}(u')$.

(II) $X_T^{\mathscr{I}}(\cdot) = X_{T'}^{\mathscr{I}}(\cdot)$ if T and T' are $A(\mathscr{I})$-conjugate.

(III) For a fixed $u \in V(A(\mathscr{I}))$, let $X_{\cdot}^{\mathscr{I}}(u)$ be the class function $w \to X_{T_w}^{\mathscr{I}}(u)$ on $W(A(\mathscr{I}))$. Then $X_{\cdot}^{\mathscr{I}}(u) - q^{2e(u)} \operatorname{sgn}_{W(A(\mathscr{I}))} \otimes \varphi_u$ is a real linear combination of $X_{\cdot}^{\mathscr{I}}(v)$'s such that $\mathrm{Cl}_{\mathbf{A}(\mathscr{I})}(v) \leqslant \mathrm{Cl}_{\mathbf{A}(\mathscr{I})}(u)$.

(IV) For $u, v \in V(A(\mathscr{I}))$, we have

$$|W(A(\mathscr{I}))|^{-1} \sum_{w \in W(A(\mathscr{I}))} (|T_w|_{q \to q^2})^{-1}\, X_{T_w}^{\mathscr{I}}(u)\, X_{T_w}^{\mathscr{I}}(v) = 0$$

if $\mathrm{Cl}_{A(\mathscr{I})}(u) \neq \mathrm{Cl}_{A(\mathscr{I})}(v)$.

It is easy to prove (I) and (II). The statement (III) follows from (3.2.1), (1.3.2)(iii), 5.2.1, and 3.1.1(iv). It remains to prove (IV). This follows from

5.4.1. Lemma. *Let T and T' be maximal tori of $A(\mathscr{I})$, and $\{u_j\}$ is a set of representatives of the unipotent conjugate classes of $A(\mathscr{I})$. Then*

$$\sum_j (\mathscr{I} - \operatorname{ind} u_j)^{-1}\, X_T^{\mathscr{I}}(u_j)\, X_{T'}^{\mathscr{I}}(v_j)$$

$$= \begin{cases} \{|A(\mathscr{I})/T|^{-1}\, |A(\mathscr{I})|_q^2\}_{q \to q^2} \cdot |N(T)/T| & \text{if } T \text{ and } T' \text{ are } A(\mathscr{I})\text{-conjugate,} \\ 0 & \text{otherwise.} \end{cases}$$

Here $N(T)$ is the normalizer of T in $A(\mathscr{I})$.

In fact, by taking the "transposition" of the orthogonality relations stated in the lemma and using the fact [17, II, 1.10] that $N(T_w)/T_w \simeq Z_{W(A(\mathscr{I}))}(w)$, we get (IV) and

$$|W(A(\mathscr{I}))|^{-1} \sum_{w \in W(A(\mathscr{I}))} |T_w|_{q \to q^2}^{-1} \cdot X_{T_w}^{\mathscr{I}}(u)^2$$

$$= \{|A(\mathscr{I})|_q^2\, |Z_{A(\mathscr{I})}(u)|^{-1}\}_{q \to q^2}$$

simultaneously. We are now going to prove 5.4.1 by induction on $\dim Z_{\mathbf{A}}(a)$. When this is minimum, i.e., when $Z_A(a) = A(\mathscr{I})$ is a maximal torus of A, this is trivially true. Now assume for the moment that q is so large that there exist characters θ and θ' of T and T' for which $\zeta_T^{A(\mathscr{I})}[\theta]$ and $\zeta_{T'}^{A(\mathscr{I})}[\theta']$ are irreducible characters up to signs. Then, by 3.3.3(i)(iv) and (2.1.3), we have

$$\sum_m (\operatorname{ind} x_m)^{-1}\, \chi_T^{\mathscr{I}}[\theta]([x_m])\, \chi_{T'}^{\mathscr{I}}[\theta']([x_m^{-1}])$$

$$= \begin{cases} |G(\mathscr{I})/K(\mathscr{I})| \cdot d^{-1} & \text{if } \zeta_T^{A(\mathscr{I})}[\theta] = \zeta_{T'}^{A(\mathscr{I})}[\theta'], \\ 0 & \text{otherwise.} \end{cases} \tag{5.4.2}$$

Here $\{x_m\}$ is a set of representatives of the conjugate classes of $A(\mathscr{I})$, and d is the degree of the irreducible constituent of $(1_{K(\mathscr{I})})^{G(\mathscr{I})}$ corresponding to the character $\chi_T^{\mathscr{I}}[\theta]$ of the Hecke algebra $\mathscr{I}=\mathscr{H}(G(\mathscr{I}), K(\mathscr{I}))$. By 4.1.1, 3.3.3, 1.2.11, and (1.2.13)(i), we have

$$d=|G(\mathscr{I})/Z_{G(\mathscr{I})}(tt^{-\tau})|_{q'},$$

where t is an element of T such that $Z_{A(\mathscr{I})}(t)=T$. Since

$$|Z_{G(\mathscr{I})}(tt^{-\tau})|_{q'}=|T|\cdot(|T|_{q\to q^2}),$$

and

$$|K(\mathscr{I})|=|A(\mathscr{I})|_{q\to q^2}\cdot q^n,$$

we see that

$$|G(\mathscr{I})/K(\mathscr{I})|\cdot d^{-1}=f(q)\,|T|, \tag{5.4.3}$$

where

$$f(q)=\{|A(\mathscr{I})/T|^{-1}\,|A(\mathscr{I})|_q^2\}_{q\to q^2}.$$

Let s be a semisimple element of $A(\mathscr{I})$, and $\{v_i\}$ be a set of representatives of the unipotent conjugate classes of $A(\mathscr{I}(s))=Z_{A(\mathscr{I})}(s)$. We can assume that $\{sv_i\}$ is a subset of $\{x_m\}$. We put

$$\sigma(s)=\sum_{\{v_i\}}(\mathscr{I}-\text{ind}\,sv_i)^{-1}\,\chi_T^{\mathscr{I}}[\theta]([sv_i])\,\chi_T^{\mathscr{I}}[\theta]([s^{-1}v_i^{-1}])).$$

If $\mathrm{Cl}_A(s)\cap T$ is empty, then this is zero since $\chi_T^{\mathscr{I}}[\theta]([sv_i])=0$ by 3.2.2. If $\mathrm{Cl}_{A(\mathscr{I})}(s)\cap T$ is non-empty, we can assume that $s\in T$, i.e., that $T\subset A(\mathscr{I}(s))$. In that case, by 3.2.2, 2.4.1, and 2.3.6, we see that $\sigma(s)$ is equal to

$$|A(\mathscr{I})/Z(s)|_{q\to q^2}^{-1}\cdot|A(\mathscr{I})/Z(s)|_q^4\sum_i(\mathscr{I}(s)-\text{ind}\,v_i)^{-1}$$

$$\left\{\sum_{x,y}X_{x^{-1}Tx}^{\mathscr{I}(s)}(v_i)\,X_{y^{-1}Ty}^{\mathscr{I}(s)}(v_i)\,\theta(xsx^{-1})\,\theta(ys^{-1}y^{-1})\right\}, \tag{5.4.4}$$

where the second sum is over $x, y\in A(\mathscr{I})/Z(s)$ such that xsx^{-1} and ysy^{-1} are contained in T. Now consider the case when $Z(s)=Z_{A(\mathscr{I})}(s)\neq A(\mathscr{I})$, i.e., when s is not in the center $Z(A(\mathscr{I}))$ of $A(\mathscr{I})$. Then, by the induction assumption,

$$\sum_i(\mathscr{I}(s)-\text{ind}\,v_i)^{-1}\,X_{x^{-1}Tx}^{\mathscr{I}(s)}(v_i)\,X_{y^{-1}Ty}^{\mathscr{I}(s)}(v_i)=0$$

unless $x^{-1}Tx$ and $y^{-1}Ty$ are $Z(s)$-conjugate. Hence the second sum in (5.4.4) can be written as

$$\sum_{x} \{X^{\mathscr{I}(s)}_{x^{-1}Tx}(v_i)^2 \sum_{\bar{n}} \theta(xsx^{-1})\, \theta^{-1}(\bar{n}xsx^{-1}\bar{n}^{-1})\},$$

where the first sum is over $x \in A(\mathscr{I})/Z(s)$ such that $xsx^{-1} \in T$, and the second one is over $\bar{n} \in N(T)/(N(T) \cap xZ(s)x^{-1})$. By using the induction assumption again, we see that $\sigma(s)$ is equal to

$$f(q)\, |(N(T) \cap Z(s))/T| \left(\sum_{x,\bar{n}} \theta(xsx^{-1})\, \theta^{-1}(\bar{n}xsx^{-1}\bar{n}^{-1}) \right)$$

if $s \notin Z(A(\mathscr{I}))$. This can be rewritten as

$$f(q) \sum_{n} \left\{ \sum_{s'} \theta(s')\, \theta^{-1}(ns'n^{-1}) \right\},$$

where the first sum is over $n \in N(T)/T$, and the second one is over $s' \in \mathrm{Cl}_{A(\mathscr{I})}(s) \cap T$. Hence, if we sum up $\sigma(s_l)$ over a set $\{s_l\}$ of representatives of the semisimple conjugate classes of $A(\mathscr{I})$, we get

$$\left\{ \sum_{s \in Z(A(\mathscr{I}))} \sigma(s) - |Z(A(\mathscr{I}))|\, f(q)\, |N(T)/T| \right\} + f(q) \sum_{n} \left\{ \sum_{t \in T} \theta(t)\, \theta^{-1}(ntn^{-1}) \right\}$$
$$= |Z(A(\mathscr{I}))|\, (\sigma(1) - f(q)\, |N(T)/T|) + f(q)\, |T|.$$

Here we have used the fact that, when $\pm\zeta_T^{A(\mathscr{I})}[\theta]$ is irreducible, $\theta^n = \theta$ ($n \in N(T)/T$) if and only if $n = 1$. On the other hand, this sum is equal to the left hand side of (5.4.2) with $T = T'$ and $\theta = \theta'$. Hence, by (5.4.2) and (5.4.3), this must be equal to $f(q)\, |T|$. Hence we get

$$\sigma(1) = f(q)\, |N(T)/T|.$$

This proves the lemma (for large q) in the case T and T' are $A(\mathscr{I})$-conjugate. If T and T' are non-conjugate, the proof can be carried out similarly and more easily. To prove the lemma for small q, we argue as follows. Let q be a power p^e of a fixed prime p. Let u and v be unipotent elements of $A(\mathscr{I})$; then, by Grothendieck's trace formula (see, e.g., [4, Appendix]), $m^{\mathscr{I}}_{u,v}$ can be written as

$$\sum_{i} \alpha_i^e - \sum_{j} \beta_j^e$$

with some complex numbers α_i, β_j depending only on the $A(\mathscr{I})$-classes of u and v. Hence, by (3.4.1), the value $X^{\mathscr{I}}_T(u)$ can also be written in a similar way, and the same can also be said for both hand sides of the equality stated in the lemma. Since we have already shown that the equality holds for large e, it must be true for any e. Hence the lemma is proved for any q. This completes the proof of Lemma 5.3.1, and hence, that of Theorem 5.3.2.

5.4.5. COROLLARY (to the proof of 5.4.3). *Let T and T' be maximal tori of $A(\mathscr{I})$, and θ and θ' be characters of T and T', respectively. Let $\{x_m\}$ be a set of representatives of conjugate classes of $A(\mathscr{I})$. Then*

$$\sum_m (\mathscr{I}-\mathrm{ind}\, x_m)^{-1} \chi_T^{\mathscr{I}}[\theta]([x_m])\, \chi_{T'}^{\mathscr{I}}[\theta']([x_m^{-1}])$$

$$=\begin{cases} \{|A(\mathscr{I})/T|^{-1}\, |A(\mathscr{I})|\, |A(\mathscr{I})|_q^2\}_{q\to q^2} \cdot |\{n\in N(T);\, \theta^n=\theta'\}| & \textit{if} \quad T=T', \\ 0 \qquad \textit{if } T \textit{ and } T' \textit{ are not } A(\mathscr{I})\textit{-conjugate.} \end{cases}$$

6. CHARACTERS OF THE HECKE ALGEBRA $\mathscr{H}(\mathrm{GL}_{2n}(\mathbf{F}_q), \mathrm{Sp}_{2n}(\mathbf{F}_q))$

Here we give a simple algorithm (6.2.2, see also its "dual" form 6.3.2) by which one can write down the characters of $\mathscr{H}$ as linear combinations of basic functions. In 6.5, we point out a connection with the theory [13] of Macdonald's symmetric functions, and use it to improve our algorithm. See 6.5.5. Combining these results with Theorem 5.3.2, we get a solution to our problem, i.e., the calculation of the character table of the Hecke algebra $\mathscr{H}$. This is summarized in Theorem 6.6.1.

6.1. Let $\eta=\pm\zeta_\Omega$ $(\Omega\in\hat{\mathscr{P}}_{2n,q})$ be an irreducible constituent of $(1_K)^G$. Then, by 4.1.1, there exists a unique element Ψ of $\hat{\mathscr{P}}_{n,q}$ such that $\Omega=2\Psi$ (see 4.2). Let χ_Ψ be the character of $\mathscr{H}$ corresponding to $\eta=\pm\zeta_{2\Psi}$, i.e.,

$$\chi_\Psi=\eta\,|\,\mathscr{H}.$$

Then, by 3.4.3, we have

$$\chi_\Psi=\sum_{\Phi\in(\Xi)} c(\Psi,\Phi)\tilde{\chi}_\Phi \tag{6.1.1}$$

with some $c(\Psi,\Phi)\in\mathbf{C}$. Here $\Xi=\Psi_s$ and $(\Xi)=\{\Phi\in\hat{\mathscr{P}}_{n,q};\, \Phi_s=\Xi\}$. Our problem is to determine the class function

$$\sum_{\Phi\in(\Xi)} c(\Psi,\Phi)\varphi_\Phi$$

on $W(\Xi)$, for any $\Xi \in \hat{\mathscr{P}}_{n,q}$ such that $\Xi = \Xi_s$. For Φ, $\Psi \in (\Xi)$, we write

$$\Phi \leqslant \Psi$$

if and only if

$$\Phi(\alpha) \leqslant \Psi(\alpha)$$

for any $\alpha \in O(L)$.

6.1.2. LEMMA. *The coefficient $c(\Psi, \Phi)$ in (6.1.1) is zero unless $\Phi \leqslant \Psi$.*

Proof. We can write

$$\tilde{\chi}_{\Phi} = \sum_{\Psi \in (\Xi)} d(\Phi, \Psi) \chi_{\Psi}$$

with $d(\Psi, \Phi) \in \mathbf{C}$. By 1.2.13 and 3.4.2, we see that $d(\Phi, \Psi) \neq 0$ implies that $2\Psi \leqslant (\Phi', \Phi')'$, i.e., that $\Psi \leqslant \Phi$. Since $(d(\Phi, \Psi))$ is the inverse matrix of $(c(\Psi, \Phi))$, we get the lemma.

6.2. For class functions ψ and φ on $W(\Xi)$, we put

$$(\psi, \varphi)_q = |W(\Xi)|^{-1} \sum_{w \in W(\Xi)} \psi(w) \varphi(w) |T_w|^{-1} |T_w|_{q \to q^2}. \tag{6.2.1}$$

6.2.2. LEMMA. (i) *For each $\Psi \in (\Xi)$, there is a unique class function d_{Ψ} on $W(\Xi)$ such that*

(ia) $d_{\Psi} = \varphi_{\Psi} + \sum_{\Phi < \Psi} e(\Psi, \Phi) \varphi_{\Phi}$, $e(\Psi, \Phi) \in \mathbf{R}$,

(ib) $(d_{\Psi}, d_{\Phi})_q = 0$ *if* $\Psi \neq \Phi$.

(ii) *We define $f(\Psi) \in \mathbf{R}$ ($\Psi \in (\Xi)$) by*

$$f(\Psi) = |W(\Xi)|^{-1} \sum_{w \in W(\Xi)} d_{\Psi}(w) \operatorname{sgn}_{W(\Xi)}(w) |T_w|^{-1}.$$

We put

$$c_{\Psi} = f(\Psi)^{-1} d_{\Psi}.$$

Then

$$\chi_{\Psi} = |W(\Xi)|^{-1} \sum_{w \in W(\Xi)} c_{\Psi}(w) \operatorname{sgn}_{W(\Xi)}(w) |T_w|^{-1} \chi^{\mathscr{H}}_{T_w}[\theta_{\Xi} | T_w]. \tag{6.2.3}$$

Proof. We define a class function c_{Ψ} on $W(\Xi)$ by (6.2.3). If Ψ and Φ

are distinct elements of (Ξ), then χ_Ψ and χ_Φ are distinct characters of H. Hence, by (2.1.3), we have

$$\sum_{\Lambda \in \mathscr{P}_{n,q}} (\operatorname{ind} a_\Lambda)^{-1} \chi_\Psi([a_\Lambda])\, \chi_\Phi([a_\Lambda^{-1}]) = 0.$$

Hence, by (6.2.3) and 5.4.5, we get

$$(c_\Psi, c_\Phi)_q = 0.$$

Moreover, by 6.1.2, we have

$$c_\Psi = \sum_{\Phi \leqslant \Psi} e'(\Psi, \Phi)\, \varphi_\Phi$$

with some $e'(\Psi, \Phi) \in \mathbf{C}$. Since $\chi_\Psi(e) = 1$, and since $\chi^{\mathscr{H}}_{T_w}[\theta](e) = 1$ for any w and θ, we also have

$$|W(\Xi)|^{-1} \sum_{w \in W(\Xi)} c_\Psi(w)\, \operatorname{sgn}_{W(\Xi)}(w)\, |T_w|^{-1} = 1.$$

Using these results, we can easily prove the existence part of (i) and (ii) simultaneously. The uniqueness part of (i) is trivial.

6.3. For class functions ψ and φ on $W(\Xi)$, we put

$$(\psi, \varphi)^*_q = |W(\Xi)|^{-1} \sum_{w \in W(\Xi)} \psi(w)\, \varphi(w)\, |T_w|\, |T_w|^{-1}_{q \to q^2}. \tag{6.3.1}$$

6.3.2. Lemma. (i) *For each $\Psi \in (\Xi)$, there is a unique class function d^*_Ψ on $W(\Xi)$ such that*

(ia) $d^*_\Psi = \varphi_\Psi + \sum_{\Phi > \Psi} e^*(\Psi, \Phi)\, \varphi_\Phi$, $e^*(\Psi, \Phi) \in \mathbf{R}$.

(ib) $(d^*_\Psi, d^*_\Phi)^*_q = 0$ *if* $\Psi \neq \Phi$.

(ii) *We define $f^*(\Psi) \in \mathbf{R}$ $(\Psi \in (\Xi))$ by*

$$f^*(\Psi) = |W(\Xi)|^{-1} \sum_{w \in W(\Xi)} d^*_\Psi(w)\, \operatorname{sgn}_{W(\Xi)}(w)\, |T_w|^{-1}_{q \to q^2}.$$

We put

$$c^*_\Psi = f^*(\Psi)^{-1}\, d^*_\Psi.$$

Then

$$\chi_\Psi = |W(\Xi)|^{-1} \sum_{w \in W(\Xi)} c^*_\Psi(w)\, \operatorname{sgn}_{W(\Xi)}(w)\, |T_w|^{-1}_{q \to q^2}\, \chi^H_{T_w}[\theta_\Xi \mid T_w]. \tag{6.3.3}$$

(iii) *For $\Psi, \Phi \in (\Xi)$, we have*

$$(c_\Psi^*, c_\Phi)_0 = \delta_{\Psi,\Phi}\, |\zeta_{2\Psi}(1)|^{-1} \prod_{i=1}^{2n} (q^i - 1),$$

where $(\cdot,\cdot)_0 = \lim_{q\to 0}(\cdot,\cdot)_q$ is the ordinary inner product for the class functions on $W(\Xi)$, and $\delta_{\Psi,\Phi}$ is the Kronecker's delta.

(iv) $f^*(\Psi)\, f(\Psi) = |\zeta_{2\Psi}(1)|\, \prod_{i=1}^{2n} (q^i - 1)^{-1}$.

(v) $c_\Psi^*(w) = |T_w|_{q\to q^2}\, |T_w|^{-1}\, c_\Psi(w),\ w \in W(\Xi)$.

Proof. For each $\Psi \in (\Xi)$, we define a class function c_Ψ^* on $W(\Xi)$ by (6.3.3). Then by (2.1.3), (6.2.3), (6.3.3), and 5.4.7, we get

$$(c_\Psi^*, c_\Phi)_0 = (c_\Psi^*, c_\Phi^*)_q = \delta_{\Psi,\Phi}\, |\zeta_{2\Psi}(1)|^{-1} \prod_{i=1}^{2n} (q^i - 1)$$

for any $\Psi, \Phi \in (\Xi)$. Parts (i)–(iii) follow from this and 6.2.2(i). Part (iv) follows from (i)–(iii) and 6.2.2, and part (v) from (6.2.3) and (6.3.3).

6.3.4. Corollary. *Let Ξ be an element of $\hat{\mathscr{P}}_{n,q}$ such that $\Xi = \Xi_s$. Then we have the following*:

(i) $\chi_{\Xi'} = |C_\Xi|_q^{-1}\, |C_\Xi|_{q'} |W(\Xi)|^{-1} \sum_{w\in W(\Xi)} |T_w|^{-1}\, \chi_{T_w}^{\mathscr{H}}[\theta_\Xi \mid T_w]$.

(ii) $\chi_\Xi = (|C_\Xi|_{q'})_{q\to q^2}\ |W(\Xi)|^{-1} \sum_{w\in W(\Xi)} \operatorname{sgn}_{W(\Xi)}(w)\ |T_w|_{q\to q^2}^{-1}\ \chi_{T_w}^{\mathscr{H}}[\theta_\Xi \mid T_w]$.

Proof. Note that Ξ' (resp. Ξ) is the unique minimal (resp. maximal) element of (Ξ). Hence (i) (resp. (ii)) follows from 6.2.3 (resp. 6.3.2) and

$$f(\Xi') = |W(\Xi)|^{-1} \sum_w |T_w|^{-1} = |C_\Xi|_q\, |C_\Xi|_{q'}^{-1}$$

(resp. $f^*(\Xi) = |W(\Xi)|^{-1} \sum_w \operatorname{sgn}_{W(\Xi)}(w)\, |T_w|_{q\to q^2}^{-1} = (|C_\Xi|_{q'}^{-1})_{q\to q^2}$).

6.3.5. *Remark.* (i) Let ζ be an irreducible character of $A = \mathrm{GL}_n(\mathbf{F}_q)$ whose degree is prime to q. Then $\zeta = \pm\zeta_\Xi$ for some $\Xi \in \hat{\mathscr{P}}_{n,q}$ such that $\Xi_s = \Xi$. By 6.3.4(ii) and 5.3.2, we have

$$\chi_\Xi([x]) = \{|\mathrm{Cl}_A(x)|\, \zeta_\Xi(x)\, \zeta_\Xi(1)^{-1}\}_{q\to q^2}$$

for any $x \in A$. Thus we have proved the conjecture mentioned in the Introduction.

(ii) In accordance with the decompositions

$$C_\Xi \simeq \prod_{\alpha\in O(L)} \mathrm{GL}_{|\Xi(\alpha)|}(\mathbf{F}_{q^{|\alpha|}}),$$

and

$$W(\Xi) \simeq \prod_{\alpha \in O(L)} S_{|\Xi(\alpha)|},$$

we have

$$d_\Psi = \bigotimes_\alpha d_{\Psi(\alpha)}(q^{|\alpha|}), \qquad d^*_\Psi = \bigotimes_\alpha d^*_{\Psi(\alpha)}(q^{|\alpha|}),$$

$$c_\Psi = \bigotimes_\alpha c_{\Psi(\alpha)}(q^{|\alpha|}), \qquad c^*_\Psi = \bigotimes_\alpha c^*_{\Psi(\alpha)}(q^{|\alpha|}).$$

Here an element λ of $\mathscr{P}$ is considered as an element of $\hat{\mathscr{P}}_{|\lambda|,q}$ by putting

$$\lambda(\alpha) = \begin{cases} \lambda & \text{if } \alpha = \{1\}, \\ (0) & \text{otherwise} \end{cases}$$

for $\alpha \in O(L)$.

(iii) By considering the limits $q \to 1$ of d_λ, for $\lambda \in \mathscr{P}_n$, we get the character table of the (commutative) Hecke algebra $\mathscr{H}(W(\mathrm{GL}_{2n}), W(\mathrm{Sp}_{2n}))$ ($W(\cdot)$ means the Weyl group). See, e.g., [12].

6.4. In Subsection 6.5, we prove some results which complement 6.2.2 and 6.3.2 using recent results of I. G. Macdonald [13] on symmetric functions. Here we review Macdonald's theory to the extent of our direct concern. Let Λ be the ring of symmetric functions in countably many independent variables $(x) = (x_1, x_2, ...)$. See [11, Chap. I]. Let r, t be independent variables, and let $F = \mathbf{Q}(r, t)$. Define an inner product $\langle \cdot, \cdot \rangle_{r,t}$ on $\Lambda_F = \Lambda \otimes_{\mathbf{Z}} F$ as follows:

$$\langle P_\lambda, P_\mu \rangle_{r,t} = \delta_{\lambda,\mu} z_\lambda \prod_{i \geq 1} \frac{1 - r^{\lambda_i}}{1 - t^{\lambda_i}}, \qquad \lambda, \mu \in P. \tag{6.4.1}$$

Here $p_\lambda = \prod_i p_{\lambda_i} = \prod_i (x_1^{\lambda_i} + x_2^{\lambda_i} + \cdots)$, $\delta_{\lambda\mu}$ is the Kronecker delta, and z_λ is the order of the centralizer of an element $w_\lambda \in S_{|\lambda|}$ of cycle type $(\lambda_1, \lambda_2, ...)$.

6.4.2. THEOREM. *For each $\lambda \in \mathscr{P}$, there is a unique symmetric function $P_\lambda = P_\lambda(x; r, t) \in \Lambda_F$, homogeneous of degree λ, and such that*

(a) *$P_\lambda = m_\lambda + \sum_{\mu < \lambda} a_{\lambda\mu} m_\mu$, $a_{\lambda\mu} \in F$, where $m_\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots + \cdots$ is the monomial symmetric function* (*see* [11]).

(b) *$\langle P_\lambda, P_\mu \rangle_{r,t} = 0$ if $\lambda \neq \mu$.*

6.4.3. DEFINITION. Let $\{Q_\lambda = Q_\lambda(x; r, t);\ \lambda \in \mathscr{P}\}$ be the basis of Λ_F dual to $\{P_\lambda\}$ with respect to $\langle \cdot, \cdot \rangle_{r,t}$.

6.4.4. THEOREM. *Let $\omega_{r,t}$ denote the F-algebra automorphism of Λ_F defined by*

$$\omega_{r,t}(p_m) = (-1)^{m-1} \frac{1-r^m}{1-t^m} p_m, \qquad m \geqslant 1.$$

Then we have

$$\omega_{r,t} P_\lambda(x; r, t) = Q_{\lambda'}(x; t, r),$$
$$\omega_{r,t} Q_\lambda(x; r, t) = P_{\lambda'}(x; t, r).$$

6.4.5. THEOREM. *Let $\lambda \in \mathscr{P}$. Let $s = (i, j)$ be a square in the diagram of λ. Define the arm-length $a(s)$ and the arm co-length $a'(s)$ of s in λ to be*

$$a(s) = \lambda_i - j, \qquad a'(s) = j-1;$$

and likewise the leg-length $l(s)$ and the leg co-length $l'(s)$ of s in λ to be

$$l(s) = \lambda'_j - i, \qquad l'(s) = i-1.$$

(i) *Let u be an indeterminate. Define an F-algebra homomorphism $\varepsilon_{u,t}\colon \Lambda_F \to F[u]$ by*

$$\varepsilon_{u,t}(p_m) = \frac{1-u^m}{1-t^m}, \qquad m \geqslant 1.$$

Then we have

$$\varepsilon_{u,t} P_\lambda(x; r, t) = \prod_{s \in \lambda} \frac{r^{a'(s)}u - t^{l'(s)}}{r^{a(s)}t^{l(s)+1} - 1}.$$

(ii) *For each $\lambda \in \mathscr{P}$ and each square s in the diagram of λ, we put*

$$b_\lambda(s) = b_\lambda(s; r, t) = \frac{1 - r^{a(s)}t^{l(s)+1}}{1 - r^{a(s)}t^{l(s)}}.$$

Then $Q_\lambda = b_\lambda P_\lambda$, where

$$b_\lambda = b_\lambda(r, t) = \prod_{s \in \lambda} b_\lambda(s).$$

6.5. We now return to the situation described in 6.1–6.3. As in 6.3.5(ii), we imbed $\mathscr{P}_n$ into $\hat{\mathscr{P}}_{n,q}$ by putting

$$\lambda(\alpha) = \begin{cases} \lambda & \text{if } \alpha = \{1\}, \\ (0) & \text{otherwise} \end{cases}$$

for $\alpha \in O(L)$. Then $\mathscr{P}_n = \{\Psi \in \hat{\mathscr{P}}_{n,q};\ \Psi_s = (n)\}$. When $\Xi = (n)$, the inner product (6.2.1) for class functions ψ, φ on $W = W(\Xi)$ is written as

$$(\psi, \varphi)_q = |W|^{-1} \sum_{w \in W} \psi(w)\, \varphi(w) \prod_i (1 + q^{\mu_i(w)}), \tag{6.5.1}$$

where $\mu(w) = (\mu_1(w), \mu_2(w), ...) \in \mathscr{P}_n$ is the cycle type of $w \in W \simeq S_n$. This suggests to us to put $r = q^2$ and $t = q$ in 6.4. In fact, we have:

6.5.2. LEMMA. *We put $E = \mathbf{Q}(q)$, considering q as a variable. Let $A(W)_E$ be the set of E-valued class functions on $W \simeq S_n$. Define an E-linear map*

$$h\colon A(W)_E \to \Lambda_E$$

by

$$h(1_{\mathrm{Cl}_W(w)}) = |Z_W(w)|^{-1}\, p_w, \qquad w \in W,$$

where we put $p_w = p_{\mu(w)}$. For $\lambda \in \mathbf{P}_n$, let d_λ and d_λ^* be the class function on W defined in 6.2.2(i) *and* 6.3.2(i), *respectively. Then we have*

$$h(d_\lambda) = P_\lambda(x; q^2, q), \tag{6.5.3}$$

and

$$\begin{aligned} h(\mathrm{sgn}_W \otimes d_\lambda^*) &= f^*(\lambda)\, f(\lambda)^{-1}\, Q_{\lambda'}(x; q, q^2) \\ &= f^*(\lambda)\, f(\lambda)^{-1}\, b_\lambda(q^2, q)^{-1}\, P_{\lambda'}(x; q, q^2). \end{aligned} \tag{6.5.4}$$

Proof. On $A(W)_E$ and Λ_E, we define scalar products $(\cdot, \cdot)_q$ and $\langle \cdot, \cdot \rangle_{q^2,q}$ by (6.5.1) and (6.4.1), respectively. It is easy to see that h preserves the scalar products. Moreover, as is well known (see [11, (7.10)]), we have

$$h(\varphi_\lambda) = s_\lambda \ (\text{: Schur function}).$$

Hence, by the definitions (6.2.2(i) and 6.4.2) of d_λ and $P_\lambda(x; q^2, q)$, for a proof of (6.5.3), it is enough to show that the transition matrix between $\{s_\lambda;\ \lambda \in \mathscr{P}_n\}$ and $\{m_\lambda;\ \lambda \in \mathscr{P}_n\}$ is "strictly upper unitriangular." But, this is again a well-known fact (see [11, (6.5)]). The automorphism $\omega_{q^2,q}$ of Λ_E defined in 6.4.4 induces an automorphism of $A(W)_E$, which we also denote by $\omega_{q^2,q}$. Then, by 6.3.2(iii), we have

$$\omega_{q^2,q}\, c_\lambda = \mathrm{sgn}_W \otimes c_\lambda^*, \qquad \lambda \in P_n.$$

Hence

$$f(\lambda)^{-1}\, \omega_{q^2,q}\, d_\lambda = f^*(\lambda)^{-1}\, \mathrm{sgn}_W \otimes d_\lambda^*.$$

By applying h on the both hand sides, we get

$$f(\lambda)^{-1}\,\omega_{q^2,q}P_\lambda(x;q^2,q)=f^*(\lambda)^{-1}\,h(\mathrm{sgn}_W\otimes d_\lambda^*).$$

Hence, (6.5.4) follows from 6.4.4.

The following result complements 6.2.2 and 6.3.2.

6.5.5. COROLLARY. *For $\lambda=(\lambda_1,\lambda_2,\ldots)\in\mathscr{P}_n$, let $f(\lambda)$ and $f^*(\lambda)$ be as in* 6.2.2(ii) *and* 6.3.2(ii), *respectively. Then we have*

$$f(\lambda)=q^{n(\lambda)}\prod_{s\in\lambda}(q^{2a(s)+l(s)+1}-1)^{-1},$$

and

$$f^*(\lambda)=q^{n(\lambda)}\prod_{s\in\lambda}(q^{2a(s)+l(s)+2}-1)^{-1},$$

where

$$n(\lambda)=\sum_i(i-1)\lambda_i.$$

(*See* 6.4.5 *for the notations $a(s)$ and $l(s)$.*)

Proof. By putting $u=0$, $r=q^2$, and $t=q$ in 6.4.5(i), we get

$$\begin{aligned}\varepsilon_{0,q}P_\lambda(x,q^2,q)&=(-1)^n\prod_{s\in\Lambda}\frac{q^{l'(s)}}{q^{2a(s)+l(s)+1}-1}\\&=(-1)^n\,q^{n(\lambda)}\prod_s(q^{2a(s)+l(s)+1}-1)^{-1}.\end{aligned}$$

On the other hand, by the definition of $\varepsilon_{0,q}$, we have

$$\varepsilon_{0,q}(p_w)=\prod_i(1-q^{\mu_i(w)})^{-1}=(-1)^n\,\mathrm{sgn}_W(w)\,|T_w|^{-1}$$

for $w\in W$. Since, by 6.5.3, we have

$$P_\lambda(x;q^2,q)=|W|^{-1}\sum_{w\in W}d_\lambda(w)\,p_w,$$

we get the formula for $f(\lambda)$ by applying $\varepsilon_{0,q}$ on both sides. The formula for $f^*(\lambda)$ follows from the one for $f(\lambda)$, (6.3.2)(iv), and

$$|\zeta_{2\lambda}(1)|=q^{n(2\lambda)}\prod_{i=1}^{2n}(q^i-1)\prod_{s\in 2\lambda}(q^{a(s)+l(s)+1}-1)^{-1}$$

(see [7]).

6.6. We now collect our main results concerning the characters of $\mathscr{H}(G, K)$ in the following.

6.6.1. THEOREM. (i) *The induced character* $(1_K)^G$ $(G=\mathrm{GL}_{2n}(\mathbf{F}_q)$, $K=\mathrm{Sp}_{2n}(\mathbf{F}_q))$ *is the sum of the irreducible characters* $\pm\zeta_{2\Psi}$ $(\Psi\in\hat{\mathscr{P}}_{n,q}$; *see* 1.2 *and* 4.2 *for the notation*) *of* G (*with multiplicity one*).

(ii) *Let* χ_Ψ $(\Psi\in\hat{\mathscr{P}}_{n,q})$ *be the irreducible character of the Hecke algebra* $\mathscr{H}(G, K)$ *corresponding to* $\pm\zeta_{2\Psi}$ *in the sense of* 2.1. *Then*

$$\chi_\Psi=|W(\Psi)|^{-1}\sum_{w\in W(\Psi)} c_\Psi(w)\,\mathrm{sgn}_{W(\Psi)}(w)\,|T_w|^{-1}\,\chi^{\mathscr{H}}_{T_w}[\theta_\Psi\,|\,T_w],$$

where $W(\Psi)\simeq\prod_{\alpha\in O(L)}S_{|\Psi(\alpha)|}$ *is the Weyl group of the subgroup* $C_\Psi\simeq\prod_\alpha \mathrm{GL}_{|\Psi(\alpha)|}(\mathbf{F}_{q^{|\alpha|}})$ *of* $\mathrm{GL}_n(\mathbf{F}_q)$, $\mathrm{sgn}_{W(\Psi)}(\cdot)$ *is the signature character of* $W(\Psi)$, T_w $(w\in W(\Psi))$ *are maximal tori of* C_Ψ, θ_Ψ *is the linear character of* C_Ψ *defined in* 1.2, $\chi^{\mathscr{H}}_{T_w}[\theta]$ *is the function on* $\mathscr{H}(G, K)$ *whose values are given in* 5.3.2, *and* c_Ψ *is the class function on* $W(\Psi)$ *defined by the following rule.*

(iia) $c_\Psi=\bigotimes_{\alpha\in O(L)} c_{\Psi(\alpha)}(q^{|\alpha|})$.

(iib) *Let* m *be a natural number. Then, for any partition* λ *of* m,

$$c_\lambda=c_\lambda(q)=q^{-n(\lambda)}\prod_{s\in\lambda}(q^{2a(s)+l(s)+1}-1)\,d_\lambda(q)$$

(*see* 6.5.5 *for the notation*). *Here* $d_\lambda=d_\lambda(q)$ $(|\lambda|=m)$ *are the class functions on* S_m *uniquely determined by the properties*

$$d_\lambda=\varphi_\lambda+\sum_{\mu<\lambda}e(\lambda,\mu)\varphi_\mu,\qquad e(\lambda,\mu)\in\mathbf{R}$$

(φ_λ $(|\lambda|=m)$ *are the irreducible characters of* S_m), *and*

$$\sum_{w\in S_m} d_\lambda(w)\,d_\nu(w)\prod_i(1+q^{\mu_i(w)})=0,\qquad \textit{if}\quad \lambda\neq\nu$$

$(\mu(w)=(\mu_1(w),\mu_2(w),...)$ *is the cycle type of* $w\in S_m$). *Equivalently,* $\{d_\lambda(w);\ |\lambda|=m,\ w\in S_m\}$ *can also be defined as the transition matrix between the symmetric functions* $p_\mu(x)=\prod_i(x_1^{\mu_i}+x_2^{\mu_i}+\cdots)$ $(x=(x_1,x_2,...)$, $\mu=(\mu_1,\mu_2,...))$ *and Macdonald's symmetric functions* $P_\lambda(x;q^2,q)$ (*See* 6.5):

$$P_\lambda(x;q^2,q)=|S_m|^{-1}\sum_{w\in S_m} d_\lambda(w)\,p_{\mu(w)}(x).$$

(iii) *Let* Ξ *be an element of* $\hat{\mathscr{P}}_{n,q}$ *such that the degree of the irreducible character* $\pm\zeta_\Xi$ *of* $\mathrm{GL}_n(\mathbf{F}_q)$ *is prime to* q. *Then*

$$\chi_\Xi([x])=\{|\mathrm{Cl}_A(x)|\,\zeta_\Xi(x)\,\zeta_\Xi(1)^{-1}\}_{q\to q^2},$$

where $A = \{(\begin{smallmatrix} 1_n & 0 \\ 0 & a \end{smallmatrix}); a \in \mathrm{GL}_n(\mathbf{F}_q)\}$ $(\simeq \mathrm{GL}_n(\mathbf{F}_q))$, x *is any element of* A, *and* $[x] = \mathrm{ind}\, x \cdot exe \in \mathscr{H}(G, K)$.

6.7. Let $G = G(q)$ be a finite group of Lie type, τ an involutive automorphism of G, and $K = G_\tau$. In view of the results of this paper and those in [2], it is natural to expect that the character theory of the Hecke algebra $\mathscr{H}(G, K)$ would be very analogous to that of finite groups of Lie type developed by Deligne, Lusztig, Springer, Kazhdan, and others (cf. [6, 4, 16]). Assuming, for simplicity, q is large, we conjecture that there should exist an irreducible character $\chi_S^{\mathscr{H}}[\theta]$ of $\mathscr{H} = \mathscr{H}(G, K)$ associated with a pair (S, θ) of a maximal τ-anisotropic torus S of G and a character θ of S such that $\theta \mid S_\tau = 1$ and

$$\{w \in N_G(S)/Z_G(S);\ \theta^w = \theta\} = \{1\}.$$

(We do not give the definition of maximal τ-anisotropic tori; but, see [14, 19] for the corresponding notion in the context of algebraic groups.) The irreducible constituent of $(1_K)^G$ corresponding to $\chi_S^{\mathscr{H}}[\theta]$ would be of degree $|G/Z_G(S)|_{q'}$. We put

$$X_S^{\mathscr{H}}(u) = \chi_S^{\mathscr{H}}[\theta]([x_u]),$$

if u is a unipotent element written as $u = x_u x_u^{-\tau}$ for some $x_u \in G$. Then we can ask if the functions $X_S^{\mathscr{H}}$ on $V(P) = \{u \in V(G); u = xx^{-\tau} \text{ for some } x \in G\}$ (see 2.3.13) have properties analogous to those satisfied by Green functions of finite groups of Lie type, e.g., independence of θ, orthogonality relations, polynomial expressions in q, relations with characters of $N_G(S)/Z_G(S)$, ... When there exists a τ-stable Borel subgroup containing a maximal τ-anisotropic torus, the situation seems to be particularly nice.

Acknowledgment

The authors thank I. G. Macdonald for kindly letting them know some of his results in [13], which were used in determining the normalizing constants $f(\lambda)$ and $f^*(\lambda)$ explicitly, before their publication.

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